Sequences

A patter or the repetition of numbers is called a sequence.
If the **difference is constant** between numbers,
then you have an **arithmetic sequence.**
Let's say you have the following arithmetic sequence.

$-8, -3, 2, 7, 12, 17 \ldots a_n$

a_n will be the term you need to find. a_1 is the first term,
d is the constant difference between two consecutive terms.

$-8, \quad -3, \quad 2, \quad 7, \quad 12, \quad 17 \ldots$ In this case $d = 5$ is the difference.

$a_1 \quad -3+5 \quad 2+5 \quad 7+5 \quad 12+5 \ldots$

You can also find d by subtracting any term by the term before.

For example:

$d = -3 - (-8) = -3 + 8 = 5$ or $\quad d = 7 - 2 = 5$

The formula to find a_n:

$$a_n = a_1 + (n-1)d$$

difference(d)

n[th] term(a_n)

first term(a_1)

Let's say you want a_{10} that is the 10[th] term.

$a_{10} = -8 + (10-1)(5) = -8 + 45 = 37$

1	2	3	4	5	6	7	8	9	10
-8,	-3,	2,	7,	12,	17,	22,	27,	32,	37

<u>Let's practice:</u> Identify if the following sequences are arithmetic and then find the 110[th] term.

(a). $-13, -5, 3, 11, 19, \ldots$

(b). $1, 3, 9, 18, 54, \ldots$

(c). $-23, -28, -33, -38, \dots$ (d). $9,15,21,27, \dots$

(e). $0, -3, -6, -9, -12, \dots$

Let's check your answers:

(a). $-13, -5, 3, 11, 19, \dots$ the first step is to identify the variables of the formula.

$a_1 = -13$ $d = 3 - (-5) = 3 + 5 = 8$ $n = 110$

$a_n = a_1 + (n-1)d$ $a_{110} = -13 + (110 - 1)(8)$

$a_{110} = 859$

(b). $1,3,9,18,54, \dots$ The first term is $a_1 = 1,$ $d = 3 - 1 = 2$ but to be an arithmetic sequence all the difference must be constant, in this case d is not constant, then this is not an arithmetic sequence.

(c). $-23, -28, -33, -38, \dots$ The first term is $a_1 = -23$
$d = -23 - (-23) = -28 + 23 = -5$, The difference is constant for all the terms, this is an arithmetic sequence

$a_n = a_1 + (n-1)d$ $a_{110} = -23 + (110 - 1)(-5) = -568$

(d). $9,15,21,27, \dots$ The first term is $a_1 = 9,$ $d = 15 - 9 = 6$
The difference is constant for all the terms, this is an arithmetic sequence

$a_n = a_1 + (n-1)d$ $a_{110} = 9 + (110 - 1)(6) = 663$

(e). $0, -3, -6, -9, -12, \dots$ The first term is $a_1 = 0,$ $d = -3 - 0 = -3$
The different is constant for all the terms, this is an arithmetic sequence.

$a_n = a_1 + (n-1)d$ $a_{110} = 0 + (110 - 1)(3) = -327$

Geometric Sequences

When your pattern has a **constant ratio**, then you have **a geometric sequence.**

For example: $-3, 6, -12, 24, -48, \ldots$

the ratio can be found by dividing any term by the term before.

$$R = \frac{6}{-3} = -2 \quad or \quad R = \frac{-48}{24} = -2$$

The formula to find the n^{th} term:

$$\boldsymbol{a_n = a_1 \,.\, R^{n-1}}$$

Let's find the 20^{th} term.

$$a_{20} = (-3)(-2)^{20-1} = (-3)(-2)^{19} = 1572864$$

Let's practice: Identify if the following sequences are geometric and then find the 10^{th} term.

(a). $-5, -35, -245, -1715$

(b). $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$

(c). $-3, -6, 12, -24, \ldots$

(d). $\frac{-3}{5}, \frac{-3}{10}, \frac{-3}{20}, \frac{-3}{40}, \ldots$

(e). $0.06, 0.36, 2.16, 12.96, \ldots$

Let's check your answers:

(a). $-5, -35, -245, -1715, \ldots$

The first term is -5. The ratio is $R = \frac{-35}{-5} = 7$

$a_n = a_1(R)^{n-1}$ $\quad a_{10} = (-5)(-7)^{10-1} = (-5)(-7)^9 = -201768035$

(b). $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{8}{3}, \ldots$ The ratio is $R = \frac{\frac{2}{3}}{\frac{1}{2}} = 2$

$a_n = a_1(R)^{n-1}$ $a_{10} = \left(\frac{1}{3}\right)(2)^{10-1} = \left(\frac{1}{3}\right)(2)^9 = \frac{512}{3}$

(c). $-3, -6, 12, -24, \ldots$ The first term is -3. The ratio is $R = \frac{-6}{-3} = 2$

But the ratio is not constant. Then this is not a geometric sequence.

(d). $\frac{-3}{5}, \frac{-3}{10}, \frac{-3}{20}, \frac{-3}{40}, \ldots$ The first term $a_1 = \frac{-3}{5}$. The ratio $R = \frac{\frac{-3}{10}}{\frac{-3}{5}} = \frac{1}{2}$

$a_n = a_1 \,.\, R^{n-1}$ $a_{10} = \left(\frac{-3}{5}\right)\left(\frac{1}{2}\right)^{10-1} = \left(\frac{-3}{5}\right)\left(\frac{1}{2}\right)^9 = \frac{-3}{2560}$

(e). $0.06, 0.36, 2.16, 12.96, \ldots$ The first term is 0.06

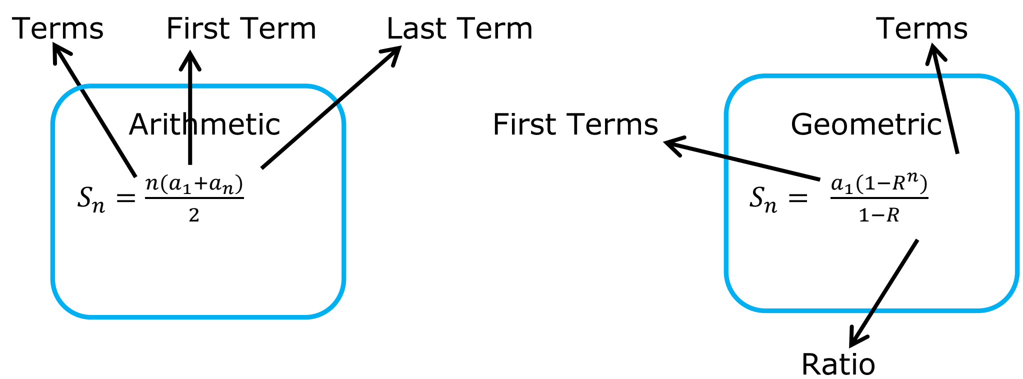
The ratio $R = \frac{0.36}{0.06} = 6$

$a_n = a_1 \,.\, R^{n-1}$ $a_{10} = (0.06)(6)^{10-1} = (0.06)(6)^9 = 604661.71$

Series (adding sequences)

Let's say you want to add either an arithmetic or geometric sequences.

The following formulas are used to find arithmetic/geometric series.



Let's do an example:

Find the sum of the first 30 terms of the following sequence.

$2, 6, 10, 14, \ldots$ The first step is to find the last term

$a_{30} = 2 + (30 - 1)(4) = 2 + 29(4) = 118$

Now you can use the formula

$S_n = \frac{n(a_1+a_n)}{2}$ $\qquad$ $S_{30} = \frac{30(2+118)}{2} = 1800$

Let's do another example:

Find the sum of the first 10 terms of the following sequence.

$-1, \frac{-5}{3}, \frac{-25}{9}, \frac{-125}{27}, \ldots$ The first step is to identify this is geometric sequence since the ratio "R" is constant. $\qquad R = \frac{\frac{-5}{3}}{-1} = \frac{5}{3}$

$$S_n = \frac{a_1(1-R^n)}{1-R} = \frac{-1(1-\left(\frac{5}{3}\right)^{10})}{1-\frac{5}{3}} = \frac{-1(1-\left(\frac{5}{3}\right)^{10})}{\frac{-2}{3}} = -246.57$$

Let's Practice:

Find the sum of the first 6 terms of the following arithmetic and geometric sequences.

(a). $600,\ 300,\ 0, \ldots$ (b). $-2, -1, -0.5, -\frac{1}{4}, \ldots$

(c). $10,15,20,25, \ldots$ (d). $-2,2,-2,2, \ldots$

Let's review your answers:

(a). $600,\ 300,\ 0, \ldots$ This is an arithmetic sequence. Let's find the 6th term

$a_6 = a_1 + (n-1)d$ $\quad a_6 = 600 + (6-1)300 = 2100$

$S_n = \frac{n(a_1+a_n)}{2}$ $\quad S_6 = \frac{6(600+2100)}{2} = 8100$

(b). $-2, -1, -0.5, -\frac{1}{4}, \ldots$ This is a geometric sequence.

Let's find ration $R = \frac{-1}{-2} = \frac{1}{2}$

$$S_n = \frac{a_1(1-R^n)}{1-R} = \frac{-2(1-\left(\frac{1}{2}\right)^6)}{1-\frac{1}{2}} = \frac{-2(1-\frac{1}{64})}{1-\frac{1}{2}} = \frac{-63}{16}$$

(c). $10,15,20,25, \ldots$ This is an arithmetic sequence. Let's find the 6th term.

$a_n = a_1 + (n-1)d = \square_6 = 10 + (6-1)(5) = 10 + 25 = 35$

$S_n = \frac{n(a_1+a_n)}{2} = S_6 = \frac{6(10+35)}{2} = 135$

(d). $-2,2,-2,2, \ldots$ This is a geometric sequence. Let's find the ratio

$R = \frac{2}{-2} = -1$

$S_n = \frac{a_1(1-R^n)}{1-R} = \frac{-2(1-(-1)^6)}{1-(-1)} = \frac{-2(1-1)}{2} = 0$

Sigma Notation

Series can also be noted using the sigma notation.

Let's do an example.

Evaluate — the last term is when $n = 6$

$\sum_{n=1}^{6} 20(3)^{n-1}$

The first term starts when $n = 1$

Sigma (means sum)

You can evaluate it by plugging the values.

$$\sum_{n=1}^{6} 20(3)^{n-1} = 20(3)^{1-1} + 20(3)^{2-1} + 20(3)^{3-1} + 20(3)^{4-1}$$

$$20(3)^{5-1} + 20(3)^{6-1}$$

$$= 20 + 60 + 180 + 540 + 1620 + 4860 = 7280$$

Let's do it using the formula. This is a geometric sequence, the $R = 3$

$S_n = \frac{a_1(1-R^n)}{1-R}$ $\quad a_1 = 20,$ $\quad a_6 = 4860,$ $\quad n = 6$

$S_6 = \frac{20(1-(3)^6)}{1-3} = 7280$

Let's do another example

$\sum_{n=1}^{13} 2n + 5$ This is an arithmetic sequence. Let's find a_1 and a_{13} to apply the formula

$a_1 = 2(1) + 5 = 7$ $\quad a_{13} = 2(13) + 5 = 31$ $\quad n = 13$

$S_n = \frac{n(a_1+a_n)}{2}$

$S_{13} = \frac{13(a_1+a_{13})}{2} \quad = \quad \frac{13(7+31)}{2} \quad = \quad 247$

Let's practice:

(a). $\sum_{n=2}^{20} -n + 5$

(b). $\sum_{n=1}^{15} -2(-3)^{n-1}$

(c). $\sum_{n=1}^{90} 3n$

(d). $\sum_{n=2}^{10} (-1)\left(\frac{1}{2}\right)^{n-1}$

Let's check your answers:

(a). -114

(b). -7174454

(c). 12285

(d). $\frac{-511}{512}$