Solving Word Problems

Developing Students'
Quantitative Reasoning Abilities

P roblem solving has been the focus of a substantial number of research studies over the past thirty years. It is well beyond the scope of this paper to even attempt to summarize this body of research. Those interested in significantly broader reviews of research related to problem solving should see Schoenfeld (1985), Charles (1987), and Charles & Silver (1988). This paper focuses on one area of research that has been of great interest to mathematics educators, solving mathematics "word problems." Some relevant research and implications for teaching are discussed in this paper.

Setting the Issue

There are many types of mathematics problems that should be included in the school curriculum. (See Charles & Lester, 1982, for a classification of mathematics problems.) This paper focuses on a particular type of problem found in the school curriculum that many teachers refer to as "word problems." Some break this type of problem into "1-step word problems" and "multiple-step word problems." Charles and Lester (1982) call this type of problem a "translation problem." Word problems have been chosen as the focus of this paper for two reasons. First, they are the most common type of problem-solving task found on most state assessments. And second, the abilities and skills related to solving word problems are key foundational abilities and skills for solving word problems in algebra.

The issue with regard to word problems is that too many students continue to be unsuccessful at solving them! Teachers still report that developing students' abilities to solve word problems is one of their most difficult and frustrating challenges. Students continue to have anxiety about solving problems, and they know that practice alone does not help them improve.

Here is a rather formal statement of what constitutes a word problem. This formal statement will be helpful when discussing implications for teaching later in this paper.

A mathematics word problem is a real-world context in which mathematical quantities are given, values of one or more quantities are known, values of one or more quantities are unknown, relationships between or among quantities are described, a question is implied or stated asking one to find the value of one or more unknown quantities, and one or more of the operations addition, subtraction, multiplication, and division can be used to find the value of the

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unknown quantity or quantities and answer the question. Data needed (known quantities) can be given in the context of the problem, in an outside data source like a graph, or created through a data collection activity. The problem might contain extraneous data, and a problem might contain one or more "hidden questions"—sub-problems that need to be solved in order to answer the question given in the problem statement. The answer to the question might be numerical (e.g., "The car costs $23,000.") or not (e.g., "He has enough money to buy the car.")

Two teaching strategies for problem solving widely used by many teachers are the key words approach and the problem-solving steps approach. If problem-solving continues to be difficult for so many teachers and too many students are not becoming successful problem solvers, one can only conclude that these common teaching strategies need to be challenged.

"The issue with regard to word problems is that too many students continue to be unsuccessful at solving them!"

Key Words: A "key words" approach teaches students to always use a particular operation whenever a word problem contains a certain English word or phrase. For example, a typical key word approach to teaching problem solving tells students to use addition whenever the question in a word problem includes "in all."

A body of research is not needed to show that a key word approach to problem solving has limited value. Most if not every state and national assessment contains word problems where a key word approach does not produce correct solutions for many problems. Sometimes "misleading" key words are used (e.g., "in all" is in the question but addition is not the needed operation), and other times problems contain no words that might be connected to a particular operation. The reason problems containing misleading or no key words are on assessments is not to set students up for failure. Rather, the fact is that most REAL problems in our world do not come neatly presented with key words telling one how to solve them. If they did, teaching problem solving would simply not be a problem! A key words approach to teaching problem solving prepares students to solve only a very small set of problems both on state assessments and in the real world.

Problem-Solving Steps: A "steps" approach to teaching problem solving gives students a sequenced set of actions to follow to solve a problem. The thinking behind this is aligned to that of teaching a skill like long division—if one follows a set of steps correctly and does the sub-calculations accurately, then one will get a correct final answer to the problem. Here is an example of "steps" for solving problems found in many instructional materials:

Step 1: Understand the problem.

Step 2: Plan a solution.

Step 3: Solve the problem.

Step 4: Check your work.

The origin of the steps approach to problem solving goes back to 1945 when George Polya, a mathematician at Stanford University, published a book on problem solving called *How to Solve It*. One the many powerful elements in that book is Polya's analysis of the phases of the problem-solving process. Polya identified four phases of solving problems.

• Understand

• Plan

• Solve

• Look Back

Polya's use of the word *phases* is crucial and gets at the heart of the issue; he did not use the word *steps* in his description of the problem-solving process because it promotes at least two misconceptions. First, "steps" suggests (like climbing stairs) that one moves off of one step and moves onto the next. This is not the way mental processing proceeds for problem solving. For example, one's understanding of a problem continues to evolve as planning and solving are underway. Another misconception that "steps" promotes, as mentioned above, is that problem solving is like a computational algorithm where there is a sequence of actions to use, which if followed correctly, will lead to the correct solution. Experience shows that problem solving is not an algorithm; there is no series of steps that guarantee success. Problem solving is a process grounded on reasoning. Certainly to be a successful problem solver there are skills needed, like reading and comprehending the words and doing the needed calculations correctly, but problem solving is not a skill.

The message for teachers that should be taken from Polya's work is that approaching problem solving in a systematic way can help students solve problems but not guarantee success. Problem-solving guides based on Polya's work like that one shown in Figure 1 can be helpful in getting students to think systematically about solving problems, but they should not be presented as "steps for finding the correct answer" for the reasons discussed above.

Figure 1: Problem-Solving Recording Sheet
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The reason so many teachers have used a key words or a steps approach to teaching problem solving is that they have not had any alternative instructional strategies. But finally, there is now a body of research that provides a new direction for teaching mathematics word problems that will produce success.

A Visual Approach to Teaching Word Problems

It was mentioned that problem solving is a process grounded on reasoning, in particular, quantitative reasoning. Quantitative reasoning involves identifying the quantities in a problem and using reasoning to identify the relationship between them. A mathematical quantity is anything that can be measured or counted. Here is a word problem whose solution requires quantitative reasoning.

Carrie has 125 U.S. stamps. She has 3 times as many foreign stamps as U.S. stamps. How many stamps does she have altogether?

The quantities in this word problem are:

• the number of U.S. stamps (a known value, 125)

• the number of foreign stamps (an unknown value)

• the total number of foreign and U.S. stamps (an unknown value)

The challenge in solving word problems is often not to identify the known and unknown quantities. Rather, the challenge is to identify statements in the problem that express relationships between quantities, to understand those relationships, and to choose an appropriate operation or operations to show those relationships. The relationships in this problem are:

• There are 3 times as many foreign stamps as U.S. stamps.

• The total for the number of foreign stamps and the U.S. stamps.

We know from research that just because a child can read a word problem, knows all vocabulary in the problem, and can identify the relationships stated in the problem it does not mean that he or she can solve it (Knifong & Holton, 1976, 1977). Rather, children who understand operation meanings and can associate relationships between quantities given in word problems with those operation meanings are better problem solvers (see Sowder, 1988). So, the challenge for teaching word problems is how to help students use quantitative reasoning—that is, use reasoning to identify the relationships between the quantities in the problem and connect those relationships to appropriate operations.

*"A new approach to solving word problems . . . is to use **bar diagrams** as visual representations . . ."*

Three research findings provide guidance for a new way to develop students' quantitative reasoning abilities.

(a) If students are encouraged to understand and meaningfully represent mathematical word problems rather than directly translate the elements of the problems into corresponding mathematical operations, they may more successfully solve these problems and better comprehend the mathematical concepts embedded within them. (Pape, S.J., 2004)

(b) Training children in the process of using diagrams to [meaningfully represent and] solve [mathematical word] problems results in more improved problem-solving performance than training students in any other strategy. (Yancey, Thompson, and Yancey, 1989)

(c) ". . . teachers need to emphasize the representation of the problem structure and de-emphasize the representation of surface features." (Diezmann and English, 2001, p. 82)

A new approach to solving word problems derived from the research findings above is to use **bar diagrams** as visual representations that show how quantities in a word problem are related. Seeing those relationships and connecting those to operation meanings enables one to select an appropriate operation for solving the problem. "A diagram can serve to 'unpack' the structure of a problem and lay the foundation for its solution" (Diezmann and English, 2001, p. 77). Nickerson (1994) found that the ability to use diagrams is integral to mathematics thinking and learning.

Here is a bar diagram representing the quantities and their relationships for the word problem given above.

U.S. stamps	⟶	125			
Foreign stamps	⟶	125	125	125	3 times as many

The relationships between the quantities in the problem can be seen in the bar diagram.

- There are 3 times as many foreign stamps as U.S. stamps.

- The set of 3 boxes that each contains 125 shows this relationship. Three equal groups are being joined.

- The total for the number of foreign stamps and the U.S. stamps.

- The combination of all four of the boxes shows this relationship. The total for the three equal groups is being joined with the amount for the one group, but since all groups have the same amount, four groups are being joined.

Translating these relationships to numerical expressions requires one to understand operation meanings. For the first relationship, three quantities are being joined. When quantities are being joined, addition can be used to find the total. But, when the quantities being joined are equal, multiplication can be used to find the total and is usually more efficient than addition. So, the numerical expression associated with the three boxes representing the number of foreign stamps is 3×125. The numerical expression that shows the joining of the number of foreign stamps and the number of U.S. stamps is $(3 \times 125) + 125$. This expression can be simplified to 4×125; the 4 groups of 125 can easily be seen in the bar diagram. The answer to the problem is that Carrie has 500 stamps altogether.

". . . a relationship in some word problems can be translated into more than one appropriate number sentence."

Figure 2 shows a collection of common 1-step word problems; each can be solved using one of the four basic operations of addition, subtraction, multiplication, or division. A bar diagram is given for each showing the relationship between the quantities. Then one or more number sentences are given showing the operation or operations that can be used to find the answer. It is important to recognize that a relationship in some word problems can be translated into more than one appropriate number sentence. For example, Example B shows that how one thinks about the relationship between the quantities in the problem leads to either an addition or subtraction number sentence; one can add on to 57 to get to 112 or one can subtract 57 from 112.

Figure 2: Bar Diagrams for Addition and Subtraction Situations

	Example A	Example B
Problem Type Joining	Kim has 23 antique dolls. Her father gives her 18 more antique dolls. Now how many antique dolls does she have?	Debbie has saved $57. How much more money does she need in order to have $112?
Diagram Showing the Relationship	$\longleftarrow$? $\longrightarrow$ | 23 | 18 |	$\longleftarrow$ 112 $\longrightarrow$ | 57 | ? |
Description of the Relationship	The two unequal parts are known and being joined and the amount in all is unknown.	The initial amount is known. The amount being joined to that is unknown. The total is known.
Number Sentence	$23 + 18 = ?$	$57 + ? = 112$ $112 - 57 = ?$

	Example C	Example D
Problem Type Separating	Steven has 122 jelly beans. He eats 71 of them in one weekend. How many jelly beans are left?	Carrie has 45 CDs. She gives some to Jo. Now Carrie has 27 left. How many did she give to Jo?
Diagram Showing the Relationship	$\longleftarrow$ 122 $\longrightarrow$ | 71 | ? |	$\longleftarrow$ 45 $\longrightarrow$ | ? | 27 |
Description of the Relationship	The total is known and the amount separated from that is known. The amount remaining is unknown.	The total is known and the amount separated from that is unknown. The amount remaining is known.
Number Sentence	$122 - 71 = ?$	$45 - ? = 27$ $27 + ? = 45$

Figure 2 (continued): Bar Diagrams for Addition and Subtraction Situations

	Example E	Example F
Problem Type **Part-Part-Whole**	Fourteen cats and 16 dogs are in the kennel. How many dogs and cats are in the kennel?	Some adults and 12 children were on a bus. There were 31 people in all on the bus. How many adults were on the bus?
Diagram Showing the Relationship	◄—— ? ——► [14 | 16]	◄—— 31 ——► [? | 12]
Description of the Relationship	Each part is known; the whole is unknown.	The first part is unknown, but the second part is known. The whole is known.
Number Sentence	14 + 16 = ?	? + 12 = 31 31 − 12 = ?

	Example G	Example H
Problem Type **Comparison**	Alex has 47 toy cars. Keisha has 12 cars. How many more cars does Alex have?	Barney has 23 old coins. Steve has 16 more old coins than Barney. How many old coins does Steve have?
Diagram Showing the Relationship	[47] [12 | ?]	[?] [23 | 16]
Description of the Relationship	Two known quantities are being compared. The amount more/less is unknown.	One quantity is known. The larger quantity is not known. The relationship of the larger quantity to the smaller quantity is given.
Number Sentence	47− 12 = ?	23 + 16 = ?

	Example I	Example J
Problem Type **Joining Equal Groups**	Kim has 4 photo albums. Each album has 85 pictures. How many photos are in her 4 albums?	Pam had 4 bags and put the same number of apples in each bag. She ended up with 52 apples in bags. How many did she put in each bag?
Diagram Showing the Relationship	←— ? —→ 85 | 85 | 85 | 85	←— 52 —→ ? | ? | ? | ?
Description of the Relationship	Four equal known amounts are being joined to find the unknown total.	A known number of unknown but equal amounts are being joined to give a known total.
Number Sentence	$4 \times 85 = ?$	$4 \times ? = 52$ $52 \div 4 = ?$

	Example K	Example L
Problem Type **Separating Equal Groups**	Byron has 45 pigeons. He keeps them in 5 pens with the same number of pigeons in each. How many pigeons are in each pen?	A total of 216 children signed up for soccer. How many 18-person teams can be made?
Diagram Showing the Relationship	←— 45 —→ ? \ ? \ ? \ ? \ ?	←— 216 —→ 18 —————→
Description of the Relationship	The total is known and being separated into a known number of equal groups, but the amount in each group is unknown.	The total is known and being separated into equal groups of a known amount. The number of equal groups needed to match the known total is unknown.
Number Sentence	$45 \div 5 = ?$	$216 \div 18 = ?$ $18 \times ? = 216$

	Example M	**Example N**
Problem Type Comparison	Alex has 17 toy cars. Keisha has 3 times as many. How many cars does Keisha have?	Barney has 24 old coins. This is 3 times more coins than Steve has. How many old coins does Steve have?
Diagram Showing the Relationship	←— 52 —→ | 17 | 17 | 17 | | 17 | | |	←— 24 —→ | ? | ? | ? | | ? | | |
Description of the Relationship	Two quantities are being compared. One is known and the other is a given number of times more. The other quantity is not known.	Two quantities are being compared. One is known and is a given number of times greater than the other. The other quantity is not known.
Number Sentence	$3 \times 17 = ?$	$3 \times ? = 24$ $24 \div 3 = ?$

For multiple-step problems such as the one above involving Carrie, multiple bar diagrams are used to help answer the hidden question (i.e., sub-problem) and then answer the final question. In the Carrie problem, the hidden question was to find the total number of foreign stamps; the answer to that was then used to answer the question stated in the problem.

One of the powerful attributes of this set of bar diagrams is that they are all connected to parts and wholes. This consistency in visual representations helps students see not only the connections between the diagrams but also connections between and among operations. An important part of understanding operations is to know all relationships between and among the four operations.

Suggestions for Teaching

Here are a few suggestions for how bar diagrams can be an integral part of teaching and learning mathematics (Diezmann, & English, 2001).

• Model bar diagrams on a regular basis; not just in special lessons but frequently when word problems are encountered.

• Discuss the structure of bar diagrams and connect them to quantities in the word problem and to operation meanings.

• Use bar diagrams to focus on the structure of a word problem, not surface features like key words.

• Encourage students to use bar diagrams to help them understand and solve problems.

Using diagrams to promote mathematical reasoning is an essential element in a student's mathematics education (NCTM, 2000). Bar diagrams can make powerful mathematics accessible to a much wider group of students; ". . . to implement the values of social justice in mathematics education, opportunities need to be provided which facilitate the development of visual literacy" (Diezmann, 1995). The bar diagrams presented here are a powerful tool for developing visual literacy and success with problem solving for ALL students.

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