1. Consider the vector space $\mathbb{R}^2$ with the two different bases

$$\{e_1, e_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \quad \text{and} \quad \{f_1, f_2\} = \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$$

Declare the basis $\{f_1, f_2\}$ to be orthonormal for the strange inner product $(\cdot, \cdot)^*$. Compute the strange inner product $(2e_1 + 3e_2, e_1 - 2e_2)^*$.

2. (Least squares as a physical problem.) Imagine several points $p_1, \ldots, p_n$ fixed in the plane. At each point p_j we attach a small Hookean spring, and connect all the springs to another movable point q. Assume all of the springs have the same spring constant, which we will call k. Thus the force on the point q due to the point p_j is $k \cdot |q - p_j|$, and directed along the line joining the two points. What we want to find is the equilibrium point for q. One way to approach this is to invoke the physical principle that says that the point q will be at rest if it is at the point of minimum energy, where the total energy of the system is given by the sums of the individual spring potential energies. Thus, the energy h of the system at the point q is given by

$$h(q) = \sum_{j=1}^{n} \frac{1}{2} k |q - p_j|^2$$

You know from calculus how to solve this problem: You just find the critical points of the function h.

(a) Find the point q as a function of the parameters p_j. What is the more common name for this point?

(b) How does the nature of the solution change if the points p_j are now allowed to be in space? (This is trivial if you use vector methods to solve for q in the first place.)

(c) (For students with a solid calculus background.) By examining the second derivative $D^2 h$, show that the point q you have found really is the minimum.

(d) What happens if the spring constants are no longer assumed to be the same? Give a geometric interpretation of this.

Many problems of maximization/minimization may be interpreted in terms of mechanical systems, and conversely. To further explore many great examples of this, check out the delightful book by Mark Levi, *The mathematical mechanic.* If you remind me, I can bring my copy to class, as you may wish to peruse it for ideas on your term essay.

3. Let an inner product on the function space $C^0[-1,1]$ be given by

$$(f,g) := \int_{-1}^{1} f(t)g(t)\,dt.$$

The problem is to produce an orthogonal set of polynomials $p_0, p_1, p_2, \ldots$ from the set $1, t, t^2, \ldots$. In class we already have computed $p_0(t) = 1$, $p_1(t) = t$, $p_2(t) = t^2 - 1/3$. Prove that if you continue in this manner, (so $p_k(t)$ is a polynomial of degree k), then the polynomials $p_{2k}(t)$ will only contain terms of even degree, and the polynomials $p_{2k+1}(t)$ will only contain terms of odd degree.

4. (a) Let $f(x) = x^2$ on $-\pi \leq x \leq \pi$. Find the Fourier series for $f(x)$.

   (b) Let $g(x) = 2x$ on $-\pi \leq x \leq \pi$. Find the Fourier series for $g(x)$ by computing the appropriate integrals.

   (c) By differentiating the series you found for f term by term, do you get the series you computed for g? Why or why not?

5. The generating function for the Legendre polynomials $P_n(x)$ is given by

$$\frac{1}{\sqrt{1-2tx+t^2}} = \sum P_n(x)t^n.$$

Show that this implies that

$$\frac{1}{2}\csc\frac{\theta}{2} = \sum_{k=0}^{\infty} P_k(\cos\theta), \quad 0 < \theta < \pi.$$