1. Let $f(z) = e^z/((1+z^2)z^2)$. Show that the residues at i and 0 are

$$\text{res}[f(z), i] = \frac{e^i}{-2i}, \quad \text{res}[f(z), 0] = 1.$$

2. Show that

$$\frac{z}{(z-1)(z-3)}$$

expanded in powers of $z-1$ is

$$\frac{-1}{2(z-1)} - 3\sum_{k=1}^{\infty} \frac{(z-1)^{k-1}}{2^{k+1}}$$

Where is this expansion valid?

3. Verify the following integral equalities

(a)

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{5 - 4\cos 2\theta}\, d\theta = \frac{3\pi}{8}.$$

(b)

$$\int_{-\pi/2}^{\pi/2} \frac{\sin\theta}{5 - 4\cos^2\theta}\, d\theta = \frac{\pi}{6}.$$

(c)

$$\int_{-\infty}^{\infty} \frac{x^2}{x^4 + x^2 + 1}\, dx = \frac{\pi}{\sqrt{3}}.$$

4. Show that

$$\int_0^{\pi} \sin^{2k}\theta\, d\theta = \frac{(2k)!}{2^{2k}(k!)^2}\,\pi, \qquad k = 1, 2, 3, \ldots$$

5. This gives an example of a calculation of a residue at an essential singularity (so the usual tricks for finding a residue do not work). Let

$$f(z) = \frac{e^{1/z}}{1-z}$$

and let us calculate the residue $\mathrm{res}[f(z), 0]$. The expansions

$$\begin{aligned} e^{1/z} &= 1 + \frac{1}{z} + \frac{1}{2!z^2} + + \frac{1}{3!z^3} \cdots \\ \frac{1}{1-z} &= 1 + z + z^2 + z^3 + \cdots \end{aligned}$$

are multiplied together to get the Laurent expansion

$$\begin{aligned} \frac{e^{1/z}}{1-z} &= (1 + z + z^2 + z^3 + \cdots)\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + + \frac{1}{3!z^3} \cdots\right) \\ &= \cdots + \frac{c_{-2}}{z^2} + \frac{c_{-1}}{z} + c_0 + c_1 z + \cdots . \end{aligned}$$

Since we are only interested in the coefficient c_{-1}, we see

$$c_{-1} = 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots$$

and we may conclude that the residue $\mathrm{res}[f(z), 0] = e - 1$.