Applied Mathematics 433 — Assignment #1

due date: October 10, 2014

1. If $z = re^{i\theta}$, $f(z) = u+iv$, where r, θ, u, v are real and $f(z)$ is an analytic function, show that the Cauchy-Riemann differential equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}.$$

2. Show that the conjugates of the complex numbers $\sin z$ and $\cos z$ are $\sin \bar{z}$ and $\cos \bar{z}$ respectively. Hence show that

$$|\sin z|^2 = \frac{1}{2}(\cosh 2y - \cos 2x), \qquad |\cos z|^2 = \frac{1}{2}(\cosh 2y + \cos 2x).$$

3. Find the first few terms of the Laurent series for

$$f(z) = \frac{1}{e^z - 1}$$

about $z = 0$. Hint: long division will do the trick ...

4. Each of the functions

$$(a)\ f(z) = \cot z \qquad (b)\ g(z) = \frac{z}{\sin z - \tan z}$$

has a pole at the origin. Find its order and residue in each case.

5. For positive numbers a and b, with $a \neq b$, show that

$$\int_{-\infty}^{+\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\, dx = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right).$$

Compute the value of the integral in the case $a = b$.

6. (Extra for experts.) An integral formula of Cauchy.

By integrating $z/(1 - ae^{-iz})$ round the rectangle with vertices at $\pm\pi$, $\pm\pi + iR$, prove that if $a \geq 1$,

$$\int_0^{\pi} \frac{ax \sin x}{1 - 2a\cos x + a^2}\, dx = \pi \log(1 + a^{-1}).$$

What is the value of the integral if $0 < a < 1$?