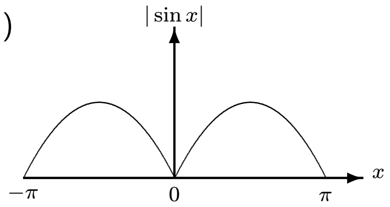
AE2 Mathematics

Solutions to Example Sheet 2: Fourier Series

1)



$f(x) = |\sin x|$ on $(-\pi, \pi)$ with $L = \pi$: $f(x)$ is an even function so $b_n = 0$. On $[0, \pi]$ we have $|\sin x| = \sin x$.

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^{\pi} |\sin x|\, dx = \frac{2}{\pi}\int_0^{\pi} \sin x\, dx \\ &= 4/\pi \end{aligned}$$

where $\cos n\pi = (-1)^n$.

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x\, \cos nx\, dx$$

We also know that $2\sin x \cos nx = \sin[(n+1)x] - \sin[(n-1)x]$ so for $n \geq 2$ (note: $a_1 = 0$)

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{\pi} \sin[(n+1)x]\, dx - \frac{1}{\pi}\int_0^{\pi} \sin[(n-1)x]\, dx \\ &= -\frac{1}{\pi}\left[\frac{\cos[(n+1)x]}{n+1}\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{\cos[(n-1)x]}{n-1}\right]_0^{\pi} \\ &= -\frac{1}{\pi}\left[(-1)^{n+1} - 1\right]\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = \frac{2\left[(-1)^{n+1} - 1\right]}{\pi(n^2-1)} \end{aligned}$$

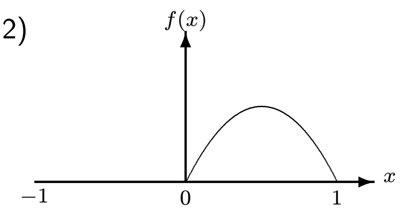
Therefore

$$a_n = \begin{cases} 0 & n = 2m+1 \quad \text{(odd)} \\ -\frac{4}{\pi(4m^2-1)} & n = 2m \quad \text{(even)} \end{cases}$$

and so

$$|\sin x| = \frac{2}{\pi} - \frac{4}{\pi}\sum_{m=1}^{\infty} \frac{\cos 2mx}{(4m^2-1)}$$

2)



$$f(x) = \begin{cases} x(1-x) & 0 \leq x \leq 1 \\ 0 & -1 \leq x \leq 0 \end{cases}$$

$L = 1$ & the function is neither odd nor even.

$$a_0 = \int_{-1}^{1} f(x)\, dx = \int_0^1 x(1-x)\, dx = 1/6$$

$$a_n = \int_0^1 x(1-x)\cos n\pi x\, dx \qquad b_n = \int_0^1 x(1-x)\sin n\pi x\, dx$$

By evaluating these integrals, one finds

$$a_n = \begin{cases} 0 & n = 2m+1 \quad \text{(odd)} \\ -\frac{1}{2m^2\pi^2} & n = 2m \quad \text{(even)} \end{cases} \qquad b_n = \begin{cases} \frac{4}{(2m+1)^3\pi^3} & n = 2m+1 \quad \text{(odd)} \\ 0 & n = 2m \quad \text{(even)} \end{cases}$$

thus giving the answer. The odd extension of $f(x) = x(1-x)$, originally defined on $0 \le x \le 1$, on the range $-1 \le x \le 1$, has $b_n = 2\int_0^1 x(1-x)\sin(n\pi x)\,dx$. Thus the answer is an odd sine-series with coefficient twice that above, namely $\frac{8}{(2m+1)^3\pi^3}$.

3) $x\sin x$ is an even function over $(-\pi, \pi)$ so $b_n = 0$ and $a_n = \frac{2}{\pi}\int_0^\pi x\sin x\cos nx\,dx$. Using the fact that $2\sin x\cos nx = \sin[(n+1)x] - \sin[(n-1)x]$, we have (except for $n = 1$)

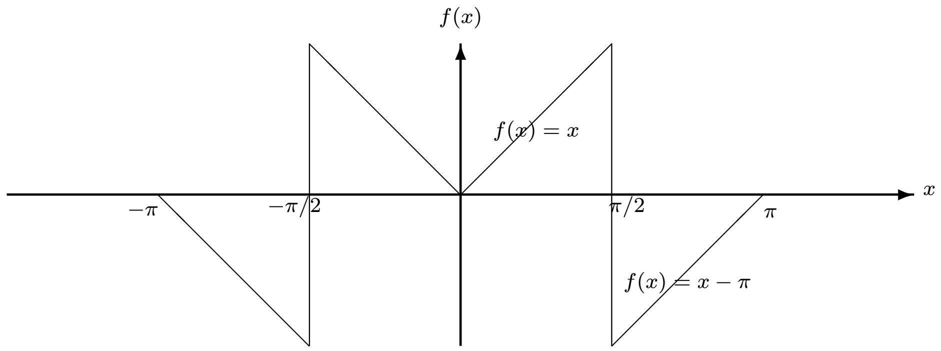
$$a_n = \frac{1}{\pi}\int_0^\pi x\sin[(n+1)x] - \sin[(n-1)x]\,dx = \frac{2(-1)^{n+1}}{n^2-1} \qquad \text{by parts}$$

Thus $a_0 = 2$ and a_1 is

$$\therefore \qquad a_1 = \frac{2}{\pi}\int_0^\pi x\sin x\cos x\,dx = \frac{1}{\pi}\int_0^\pi x\sin 2x\,dx = -\tfrac{1}{2} \qquad \text{(by parts)}$$

$$x\sin x = 1 - \tfrac{1}{2}\cos x + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos nx$$

4)



The figure shows that $f(x)$ is even about $x = 0$; thus $L = \pi$ and $b_n = 0$ and

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi}(x-\pi)\,dx\right\} = 0$$

$$\begin{aligned} a_n &= \frac{2}{\pi}\left\{\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi}(x-\pi)\cos nx\,dx\right\} \\ &= \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx - 2\int_{\pi/2}^{\pi}\cos nx\,dx \end{aligned}$$

By parts

$$\int_0^\pi x\cos nx\,dx = \frac{1}{n^2}\left[nx\sin nx + \cos nx\right]_0^\pi = \frac{(-1)^n - 1}{n^2} = -\frac{2}{(2m+1)^2}$$

when $n = 2m+1$ and zero when n is even. Moreover,

$$\int_{\pi/2}^{\pi}\cos nx\,dx = \frac{1}{n}\left[\sin nx\right]_{\pi/2}^{\pi} = -\frac{\sin\frac{1}{2}n\pi}{n} = -\frac{(-1)^m}{2m+1}$$

thus giving the answer as advertised.