Solution to Systems of Linear Equations

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Systems of Linear Equations

- A system of m simultaneous linear equations in m unknowns $x_1, x_2, x_3, \ldots, x_m$ is of the form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1m}x_m &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{2m}x_m &= b_2 \\
\ldots\ldots \qquad \ldots \qquad\qquad \ldots \qquad &\ \ \ldots \\
a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mm}x_m &= b_m
\end{aligned}
$$

- Where the coefficients a_{ij} and b_j are numbers.

Systems of Linear Equations

- Matrix form of the above set of simultaneous equations can be written as $AX = B$,where $A_{m\times m}$ is the coefficient matrix,

  $X_{m\times 1}$ is a column matrix of unknowns

  and B is a column matrix of constants

Systems of Linear Equations

- Solution of simultaneous equation is meant to compute the numeric values of $x_i\left(i=1,2,3,\ldots,m\right)$ which satisfies all the equations of the set.

- If all the values of b_j are zero, then the system of simultaneous equations is said to be homogeneous, otherwise non-homogeneous.

Systems of Linear Equations

- E.g., of a homogeneous equation:

$$4x_1 + 3x_2 - 5x_3 = 0$$
$$2x_1 - 4x_2 + 3x_3 = 0$$
$$x_1 + x_2 - 3x_3 = 0$$

- E.g., of non homogeneous equation:

$$x_1 + 2x_2 + 3x_3 = 11$$
$$2x_1 + 3x_2 - 6x_3 = 7$$
$$4x_1 - 3x_2 7x_3 = 3$$

Systems of Linear Equations

- As stated earlier, a set of simultaneous equation can be expressed as $AX = B$ in matrix form.
- The solution matrix X can be obtained by the equation

$$X = A^{-1}B$$

- But this method is impractical for large systems even with efficient ways of computing the determinants

Back Substitution

- The back substitution is an algorithm, which is useful for solving a linear system of equations that has an upper-triangular coefficient matrix
- **Definition- Upper-Triangular Matrix**

  An nxn matrix is called ***upper triangular*** provided that the elements satisfy $a_{ij} = 0$ whenever i > j. i.e., all entries below main triangular are zero.

Back Substitution

- If **A is an upper-triangular matrix, then** AX = B is said to be an upper-triangular system of linear equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n &= b_1 \\
a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n &= b_2 \\
a_{33}x_3 + \ldots + a_{3n}x_3 &= b_3 \\
\ldots \qquad \ldots \qquad & \ldots \\
a_{nn}x_n &= b_n
\end{aligned}
$$

Back Substitution

- **Theorem (Back Substitution).**

  Suppose that AX=B is an upper-triangular system with the form given above in (1).

  If $a_{ii} \neq 0$ for $i = 1, 2, \ldots, n$ then there exists a unique solution.

Example 1

- Use the back-substitution method to solve the upper-triangular linear system

$$\begin{pmatrix} 4 & -1 & 2 & 3 \\ 0 & -2 & 7 & -4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ -7 \\ 4 \\ 6 \end{pmatrix}$$

Example 2

- Use the back-substitution method to solve the upper-triangular linear system

$$\begin{pmatrix} 4 & -1 & 2 & 3 \\ 0 & -7 & 6 & -4 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \\ -12 \\ -8 \end{pmatrix}$$

Gauss Elimination Method

- By using ERO, matrix A is transformed into an upper triangular matrix (all elements below diagonal is 0)
- Back substitution is used to solve the upper-triangular system

$$\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \underset{\text{ERO}}{\Rightarrow} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \tilde{a}_{ii} & \cdots & \tilde{a}_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

Back substitution

Gauss Elimination Method

- At the end of ERO, we must arrive at :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a_{22}^{'} & a_{23}^{'} & \cdots & a_{2n}^{'} \\ & & a_{33}^{''} & \cdots & a_{3n}^{''} \\ & & & \vdots & \vdots \\ & & & & a_{nn}^{(n-1)} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2^{'} \\ b_3^{''} \\ \vdots \\ b_n^{(n-1)} \end{bmatrix}$$

Example 1

- Solve the following system of simultaneous equations

$$3x_1 + 4x_2 + 5x_3 = 4$$

$$6x_1 + 2x_2 + 3x_3 = 7$$

$$x_1 + 3x_2 + 3x_3 = 1$$

Using the Gaussian elimination method

Solution