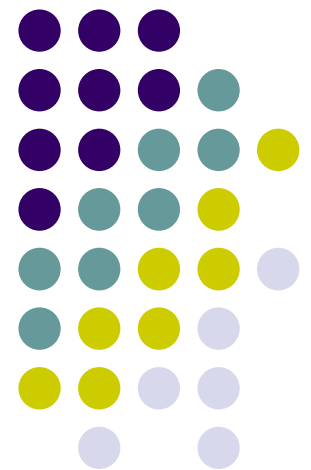


Solution to Systems of Linear Equations

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Systems of Linear Equations

- A system of **m** simultaneous linear equations in **m** unknowns $x_1, x_2, x_3, \dots, x_m$ is of the form

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1m}x_m = b_1$$

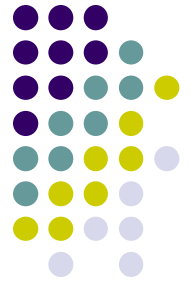
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2m}x_m = b_2$$

$$\dots \dots \dots \dots \dots \dots$$

$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mm}x_m = b_m$$

- Where the coefficients a_{ij} and b_j are numbers.

Systems of Linear Equations



- Matrix form of the above set of simultaneous equations can be written as $AX = B$, where $A_{m \times m}$ is the coefficient matrix,

$X_{m \times 1}$ is a column matrix of unknowns
and B is a column matrix of constants

Systems of Linear Equations



- Solution of simultaneous equation is meant to compute the numeric values of $x_i (i = 1, 2, 3, \dots, m)$ which satisfies all the equations of the set.
- If all the values of b_j are zero, then the system of simultaneous equations is said to be **homogeneous**, otherwise **non-homogeneous**.



Systems of Linear Equations

- E.g., of a homogeneous equation:

$$4x_1 + 3x_2 - 5x_3 = 0$$

$$2x_1 - 4x_2 + 3x_3 = 0$$

$$x_1 + x_2 - 3x_3 = 0$$

- E.g., of non homogeneous equation:

$$x_1 + 2x_2 + 3x_3 = 11$$

$$2x_1 + 3x_2 - 6x_3 = 7$$

$$4x_1 - 3x_2 + 7x_3 = 3$$



Systems of Linear Equations

- As stated earlier, a set of simultaneous equation can be expressed as $AX = B$ in matrix form.
- The solution matrix X can be obtained by the equation

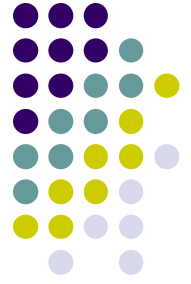
$$X = A^{-1}B$$

- But this method is impractical for large systems even with efficient ways of computing the determinants



Back Substitution

- The **back substitution** is an algorithm, which is useful for solving a linear system of equations that has an upper-triangular coefficient matrix
- **Definition- Upper-Triangular Matrix**
An **$n \times n$** matrix is called ***upper triangular*** provided that the elements satisfy $a_{ij} = 0$ whenever $i > j$. i.e., all entries below main triangular are zero.



Back Substitution

- If **A** is an upper-triangular matrix, then $AX = B$ is said to be an upper-triangular system of linear equations

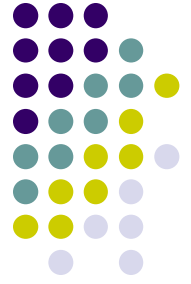
$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1$$

$$a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2$$

$$a_{33}x_3 + \dots + a_{3n}x_n = b_3$$

$$\dots \quad \dots \quad \dots$$

$$a_{nn}x_n = b_n$$



Back Substitution

- **Theorem (Back Substitution).**

Suppose that $AX=B$ is an upper-triangular system with the form given above in (1).

If $a_{ii} \neq 0$ for $i = 1, 2, \dots, n$ then there exists a unique solution.



Example 1

- Use the back-substitution method to solve the upper-triangular linear system

$$\begin{pmatrix} 4 & -1 & 2 & 3 \\ 0 & -2 & 7 & -4 \\ 0 & 0 & 6 & 5 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 20 \\ -7 \\ 4 \\ 6 \end{pmatrix}$$



Example 2

- Use the back-substitution method to solve the upper-triangular linear system

$$\begin{pmatrix} 4 & -1 & 2 & 3 \\ 0 & -7 & 6 & -4 \\ 0 & 0 & -2 & 5 \\ 0 & 0 & 0 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 25 \\ 6 \\ -12 \\ -8 \end{pmatrix}$$

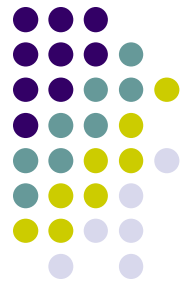


Gauss Elimination Method

- By using ERO, matrix A is transformed into an upper triangular matrix (all elements below diagonal is 0)
- Back substitution is used to solve the upper-triangular system

$$\begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ii} & \cdots & a_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{ni} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_n \end{bmatrix} \xRightarrow{\text{ERO}} \begin{bmatrix} a_{11} & \cdots & a_{1i} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots & & \vdots \\ 0 & \cdots & \tilde{a}_{ii} & \cdots & \tilde{a}_{in} \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & \tilde{a}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ \tilde{b}_i \\ \vdots \\ \tilde{b}_n \end{bmatrix}$$

Back substitution



Gauss Elimination Method

- At the end of ERO, we must arrive at :

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ & a'_{22} & a'_{23} & \cdots & a'_{2n} \\ & & a''_{33} & \cdots & a''_{3n} \\ & & & \ddots & \vdots \\ & & & & a^{(n-1)}_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b'_2 \\ b''_3 \\ \vdots \\ b^{(n-1)}_n \end{bmatrix}$$



Example 1

- Solve the following system of simultaneous equations

$$3x_1 + 4x_2 + 5x_3 = 4$$

$$6x_1 + 2x_2 + 3x_3 = 7$$

$$x_1 + 3x_2 + 3x_3 = 1$$

Using the Gaussian elimination method

Solution

