1 Coordinate Geometry

Keywords:

point

coordinates

distance

line

line segment

slope

parallel

circle

center

radius.

Skills:

A. 1. Given a point in the coordinate plane, obtain its coordinates.
2. Given the coordinates of a point, locate it in the coordinate plane.

B. Given two points, find
1. The distance between them;
2. The slope of the line segment connecting them.

C. Given the equation of a straight line, sketch its graph. Given a drawing of a straight line in a coordinate plane, find its equation (approximately).

D. Given a point and the slope or given two points of a line, find the equation of the line.

E. Given the equations of two straight lines,
1. determine whether they are parallel;
2. if not, find their intersection.

F. Solve problems involving straight lines using coordinate methods.

G. Obtain the equation of a circle given its center point and radius. Given an equation of a circle, find its center and radius.

1.1 The Coordinate Plane. *Use Plate 1-1*

Coordinate geometry is an algebraic treatment of geometry. It is based on the representation of any point in space by an ordered set of numbers.

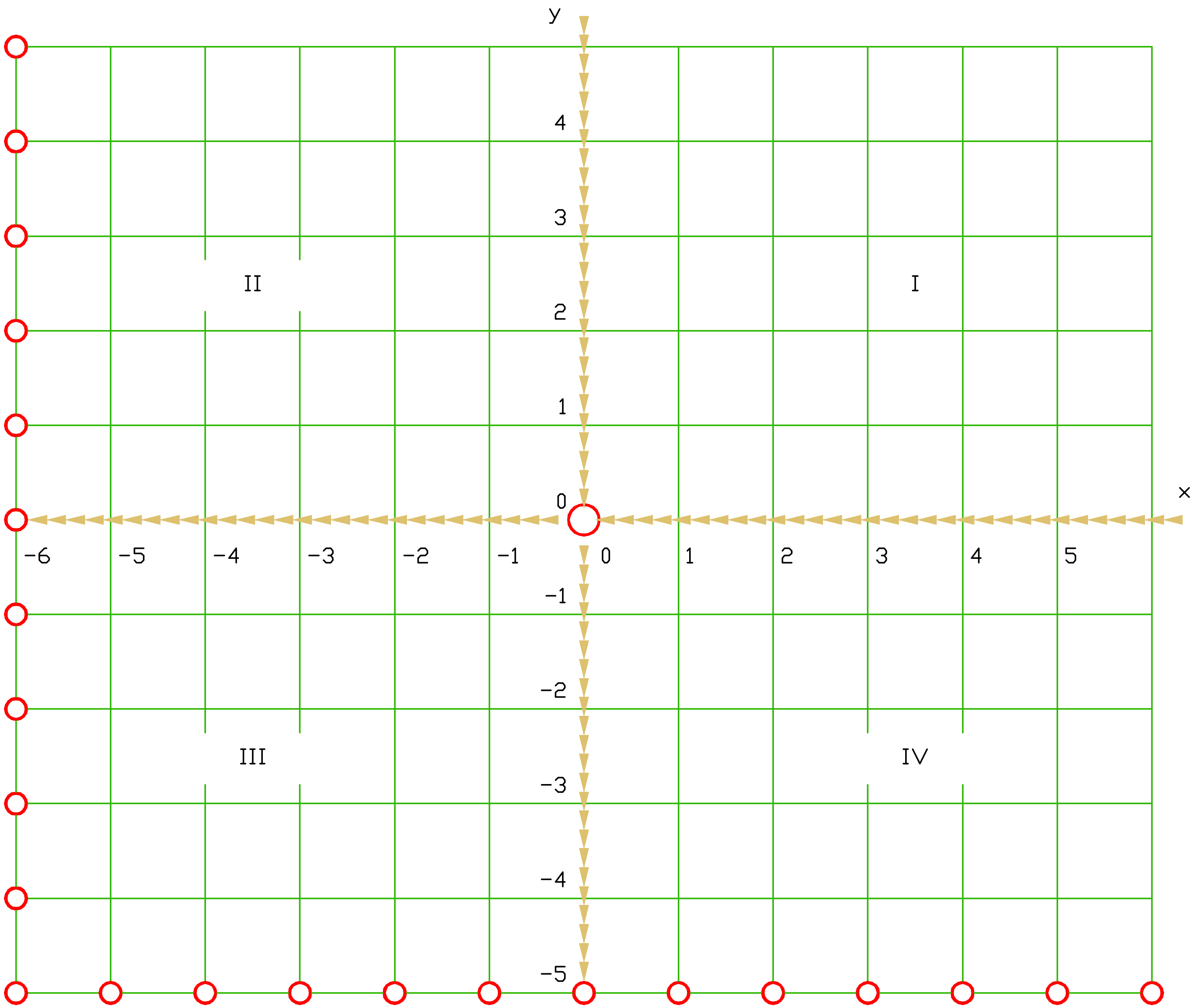
In this course, most of our work will deal with the *coordinate plane*. This can be thought of as a Euclidean plane containing a selected pair of perpendicular number lines called *axes*. The first of these lines, stretching horizontally from left to right, is called the *x-axis* and the zero on this line is called the *origin*. Positive numbers on this axis are represented by distances measured from the origin to the right, negative numbers, to the left. The other axis, the *y-axis*, is a vertical number line passing through the origin and having its zero there. Positive numbers on the y-axis are represented by distances measured upward from the origin, negative numbers, downward.

{*Graphics note: Locate the origin in Figure 1-1. The axes are bumpy lines that cross at the origin. These lines feel smoother as a finger glides along them in the positive direction, rougher, in the negative direction.*}

The axes divide the plane into four *quadrants* which are labeled by Roman numerals, I, II, III, IV, in *counterclockwise* order circling around the origin starting upward from the positive x-axis.

{*Graphics note: To locate the quadrants in counterclockwise order, proceed as follows. At each step, press to get a message confirming your location. First find the origin. Then, go to the right to a point on the positive x-axis. Proceed upward into the first quadrant. Then, go left to the positive y-axis. Go further left into the second quadrant. Then, go down to the negative x-axis. Go further down into the third quadrant. From there, go right to the negative y-axis. Go further right into the fourth quadrant. Upward from there lies the positive x-axis, from which the process can begin again.*}

We identify a point P in the coordinate plane by a pair of numbers, its *coordinates*, (a, b). The first coordinate, a, is the horizontal *displacement* of P from the y-axis, that is, the distance of P from the y-axis with an attached sign, positive if P lies to the right of the axis, negative, if to the left. This may also be thought of as the



KEYWORDS

- origin
- x-axis
- y-axis
- coordinate
- abscissa
- ordinate
- quadrant
- clockwise
- counter-clockwise

Fig. 1-1

Figure 1-1

number on the x-axis at the foot of the perpendicular from P to the x-axis. It is called the *x-coordinate* or *abscissa* of P. Similarly, the *y-coordinate* or *ordinate* of P is the number on the y-axis corresponding to the foot of the perpendicular from P to the y-axis. The ordinate of P is the vertical displacement of P, from the x-axis, positive if P lies above the axis, negative, if below.
{*Note: Nemeth Braille for the point* (a, b) *is left parenthesis, a, math comma, space, b, right parenthesis.*}

EXERCISES 1.1

1. a. Locate the origin.

b. Locate the x-axis. {*Graphics note: Slide to the right or left of the origin along the embossed line and press at any point.*}

c. Locate the y-axis.

d. Locate the four quadrants.

e. Find the integer points on the x-axis, $x = 0, 1, 2, 3, \ldots, -1, -2, -3, \ldots$. Do the same for the y-axis.

{*Graphics note: In addition to the cues given by the voiced x-coordinate buttons along the bottom edge of the graph, each integer point on the x-axis is labeled in Braille just below the x-axis immediately to the right of the corresponding point. Similarly, each integer point on the y-axis has a Braille label immediately to the left of the y-axis above the point.*}

2. For each of the following points, name the quadrant which contains it:
a. $(1, 5)$ b. $(1, -5)$ c. $(-5, 1)$ d. $(-3, -2)$

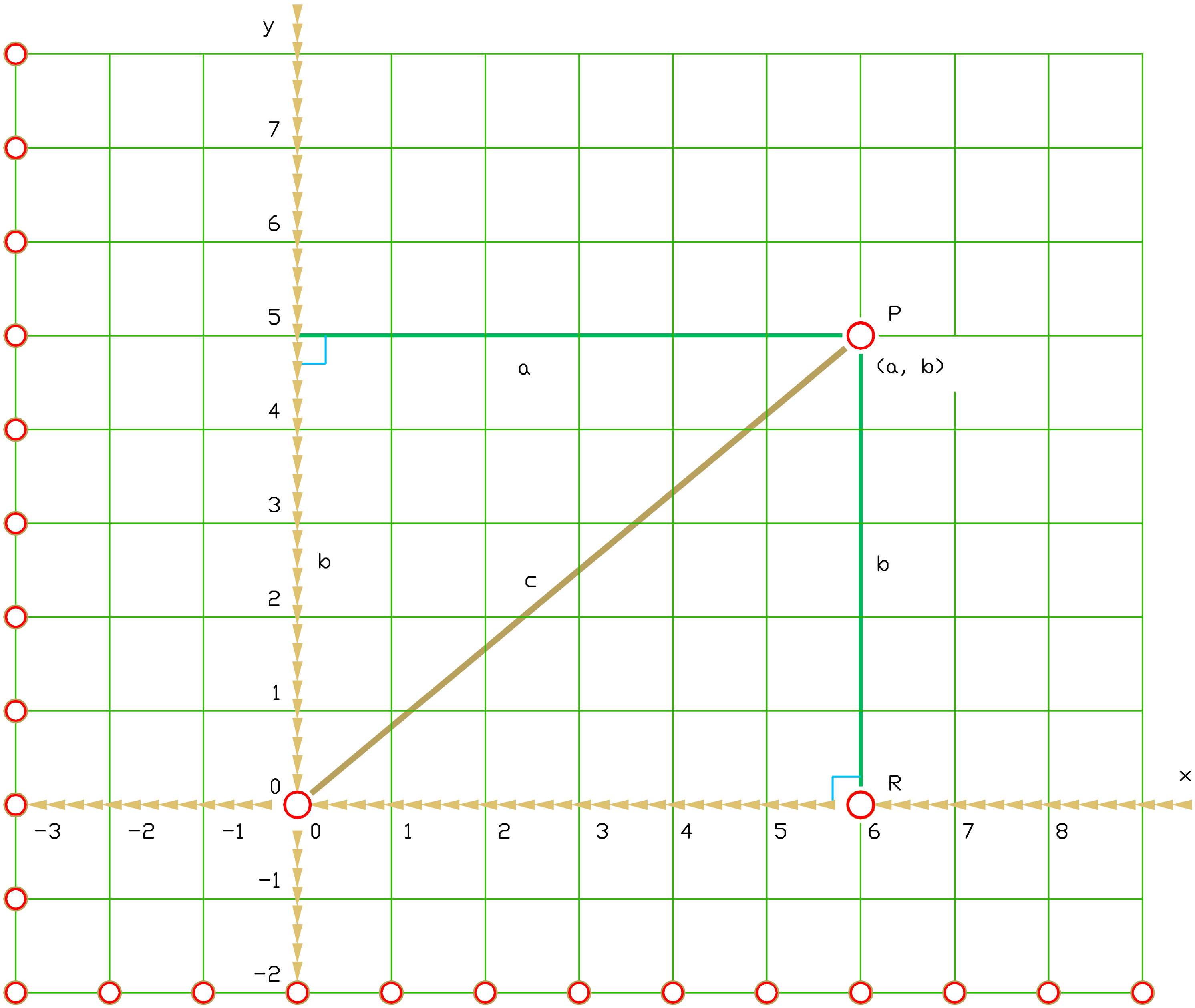
3. Locate approximately the points
a. $(2, 1)$ b. $(-2, 1.5)$ c. $(-1, -2.5)$ d. $(-1.25, -1.75)$

4. Locate the points P, Q, R {*their names will sound when the corresponding button is pressed*} and estimate their coordinates.

1.2 Distance. (*Use Plate 1-2.*)

The formula for the distance OP of a point $P = (a, b)$ from the origin $O = (0, 0)$ is very useful. Let R be the foot of the perpendicular from P to the x-axis. The triangle $\triangle ORP$ is a right triangle with the hypotenuse OP, the horizontal leg OR, and the vertical leg RP. To calculate the distance OP, we use the Pythagorean Theorem: for a right triangle with legs, a and b, and hypotenuse, c,

$$c^2 = a^2 + b^2.$$



KEYWORDS
- origin
- x axis
- y axis
- coordinate
- abscissa
- ordinate
- hypotenuse
- length
- displacement

Figure 1-2

Figure 1-2

As shown in Figure 1-2, the length of OR is $|a|$, the absolute value of the x-coordinate of P and, similarly, the length of RP is $|b|$, the absolute value of the y-coordinate. From the Pythagorean Theorem, the length of OP is

$$c = \sqrt{|a|^2 + |b|^2} = \sqrt{a^2 + b^2}$$

If, for example, $P = (3, 4)$, the foot of the perpendicular from P to the x-axis is $R = (3, 0)$, $a = 3$ and $b = 4$. The distance of P from the origin is

$$c = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

In this example, $\triangle ORP$ is the 3, 4, 5 right triangle that may be familiar to you from elementary geometry.

EXERCISES 1.2A

{*Graphics note: Leave the plate, Figure 1-2, on the touch pad and select the audio file, Fig1-2A.*}

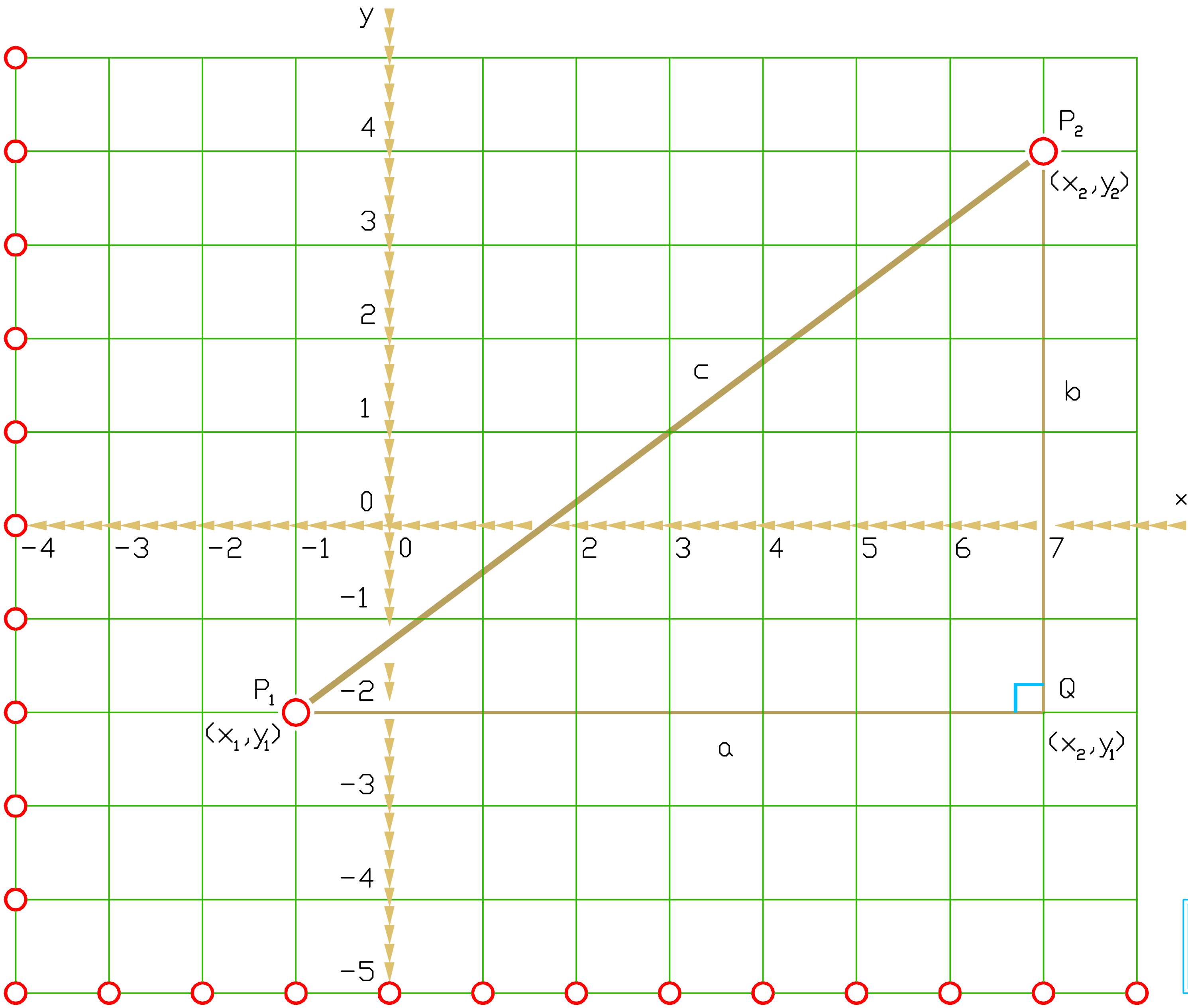
1a. With a Braille ruler, measure the distance from the origin of each of the following points.
(i). $(4, 3)$
(ii). $(6, 8)$
(iii). $(1, 1)$
(iv). $(-3, 3)$
(v). $(-1, \frac{3}{4})$
(vi). $(-3, -2)$
(vii). $(1, \sqrt{2})$

b. For each of these points, calculate the distance, d, from the origin using the Pythagorean Theorem and compare that value with the measured result.

2. Locate the points P, Q, R, S, T, U, in Figure 1-2A and calculate their distances from the origin by the Pythagorean Theorem.

{*Note: As in the next paragraph, we often use symbols with subscripts attached, for example,* x_1. *This is spoken as 'x sub 1' or, briefly, as 'x 1'.*}

The Pythagorean Theorem can be used similarly to calculate the distance between any two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ in the coordinate plane.
{*Graphics note: Put the plate, Figure 1-3, on touch pad and select the audio file, Fig1-3.*}
In Figure 1-3, we construct a right triangle P_1QP_2 with the right angle at the intersection $Q = (x_2, y_1)$ of the horizontal line through P_1 and the vertical line through



KEYWORDS ◇
displacement ◇
horizontal ◇
vertical ◇
run ◇
rise ◇
length ◇
distance ◇

Fig. 1-3

Figure 1-3

P_2. The hypotenuse of the right triangle is the line segment, P_1P_2, and the legs are the horizontal segment, P_1Q, and the vertical segment, QP_2. Sometimes the right triangle is *degenerate*, for example, when $x_1 = x_2$. In that case, we say the horizontal leg has zero length. Similarly, when $y_1 = y_2$, we say the vertical leg has zero length.

The quantity, $x_2 - x_1$, is called the *horizontal displacement* from P_1 to Q; it is positive if Q lies to the right of P_1, negative, if left. This is also called the *run* from P_1 to P_2. Similarly, the *vertical displacement* from Q to P_2 is $y_2 - y_1$; it is positive if P_2 lies above Q, negative, if below. This is also called the *rise* from P_1 to P_2. Since distance is never negative, the lengths of the legs of the right triangle are the absolute values of these displacements:

$$a = |x_2 - x_1| \quad \text{and} \quad b = |y_2 - y_1|$$

It follows from the Pythagorean Theorem that the hypotenuse or the distance from P_1 to P_2 is

$$c = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Notice that the absolute value signs need not appear in this formula, since squares are always nonnegative.

EXERCISES 1.2B

{*Graphics note: Leave the plate, Figure 1-2, on the touch pad and select the audio file, Fig1-2B.*}

1. Calculate the distance between the two points for each of the following pairs.

a. $(0,0)$ and $(3,4)$ b. $(1,0)$ and $(0,1)$
c. $(1,1)$ and $(4,-3)$ d. $(-1,2)$ and $(-2,1)$
e. $(\frac{1}{2},1)$ and $(-1,-1)$ f. $(\frac{3}{2},0)$ and $(1,\frac{-\sqrt{3}}{2})$

2. For each of the pairs in Ex. 1, what do the changes in x and y from the first point to the second tell about the relative positions of the two points?
For example, in Part a, the point $(3,4)$ *is three units to the right and four units above the point* $(0,0)$.

3. Locate each of the points A, B, C in the graphic Fig1-2B, estimate their coordinates, and use the values to calculate the lengths of the sides of the triangle ABC.

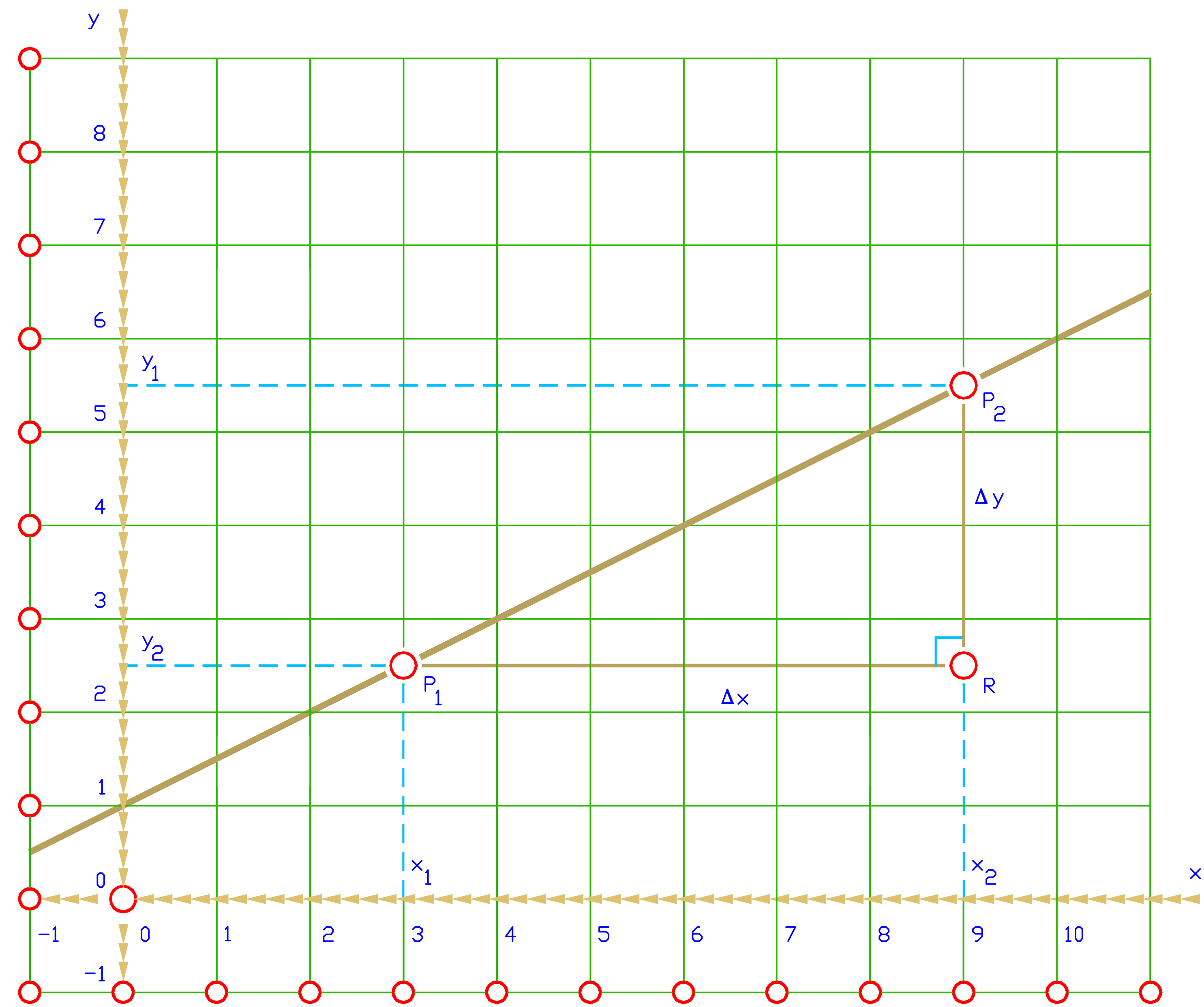
1.2.1 Delta Notation.

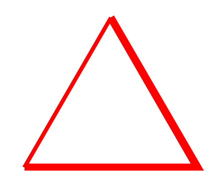
{*Use Plate 1-4.*}

For the two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$, the run or the *change in* x and the rise or the *change in* y are often denoted by the respective symbols

$$\Delta x = x_2 - x_1 \qquad \text{and} \qquad \Delta y = y_2 - y_1,$$

KEYWORDS
line
run
rise
slope





Greek capital delta

Fig. 1-4

Figure 1-4

where Δ is the Greek capital letter corresponding to the Roman D.

{*Graphics note: An enlarged representation of the print symbol,* Δ *is embossed on Plate 1-4, above the plate identification box. The form of* Δ *is an equilateral triangle on a horizontal base. In the Roman font, the left leg is thinner than the other two sides of the triangle. The Nemeth Braille code for* Δ *consists of three cells: Greek letter (dots 4 6), capital (dot 6), d (dots 1 4 5)*}

In Delta notation, the formula for the distance from P_1 to P_2 becomes

$$c = \sqrt{\Delta x^2 + \Delta y^2}$$

1.3 Straight Lines.

{*Use Plates 1-4 – 1-8.*}

A straight line in the coordinate plane is the set of all points (x, y) that satisfy a *linear* equation,

$$ax + by + c = 0,$$

where the coefficients a, b, and c are constants and the coefficients of x and y are not both zero. We have already seen two important special cases, the horizontal and vertical lines of a coordinate grid: the vertical lines have equations in the form

$$x - p = 0,$$

where on each line, x has a constant coordinate, p, and y varies over all real values. For example, when $p = 0$, the equation becomes $x = 0$ and the vertical line is the y-axis. Similarly, the horizontal lines have equations in the form

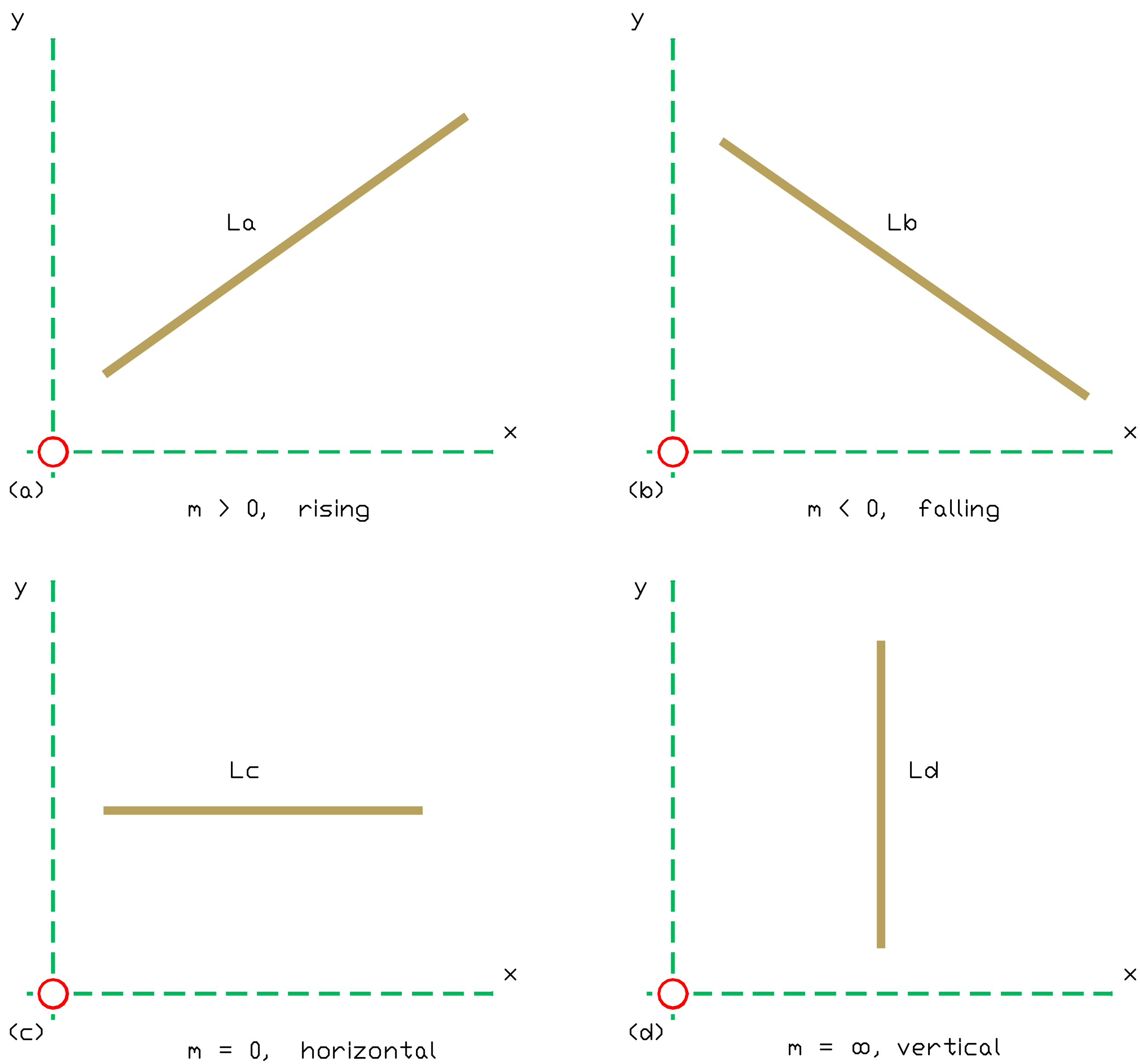
$$y - q = 0$$

where q is the constant y-coordinate on a given horizontal line and x is unconstrained. When $q = 0$, this equation describes the x-axis.

Figure 1-4 displays the straight line that satisfies the equation

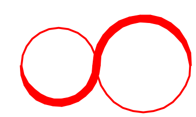
$$x - 2y + 2 = 0.$$

If the constants in this equation are all multiplied by the same nonzero number, we can get another equation for the line in the same form, for example, multiplying by 2 in the preceding equation, we get $2x - 4y + 4 = 0$. So, there can be no fixed geometrical interpretation for any one of these constants by itself. We now offer several forms of the equation for a straight line in which the constants do have unique geometrical meanings.



KEYWORDS

- straight line
- rise
- run
- slope
- rising
- falling
- horizontal
- vertical
- infinity



infinity symbol

Fig. 1-5

Figure 1-5

Slope. {*Use Plate 1-4.*} Figure 1-4 shows two points P_1 and P_2 on the line and a third point, R displaced horizontally from P_1 by the amount Δx. Similarly, P_2 is displaced vertically from R by the amount Δy. The triangle P_1RP_2 is a right triangle with a horizontal leg $\overline{P_1R}$ and vertical leg $\overline{RP_2}$. All triangles formed in this way from any two points of the straight line are similar. It follows that the ratio of the rise to the run,

$$m = \frac{\Delta y}{\Delta x},$$

is a constant independent of the choice of the two points, P_1 and P_2. The constant, m, is called the *slope* of the line. The slope is a measure of the direction of the line.

For the line in Figure 1-4, the slope is $m = \frac{1}{2}$.

The sign of the slope has a geometrical meaning shown by the four sketches in Figure 1-5:

In Sketch a, the slope is positive, hence, the rise and the run have the same sign. Reading from left to right, as x increases, positive slope implies that y increases. In this case, we say the line is *rising*.

In Sketch b, the slope is negative, hence, the rise and the run have opposite signs. Reading from left to right, as x increases, negative slope implies that y decreases. In this case, we say the line is *falling*.

In sketch c, the slope is zero. The rise is zero for every run. The line is horizontal and y has the same value for all x.

In sketch d, the line is vertical. Here, x has the same value for all y. The run between any two points on the line is 0. The slope is undefined. It is sometimes said that the slope, m, of a vertical line is *infinity*, written, $m = \infty$. This means only that the line is vertical. It doesn't define a slope. It expresses the behavior of the slope of a line being rotated toward a vertical orientation. As the line approaches the vertical, its slope, in the absolute sense, becomes large beyond any preset bound. The slope of a nearly vertical line could be either large and negative or large and positive. {*Graphics note: Nemeth code for the infinity symbol is capital (dot 6) full cell (dots 1 2 3 4 5 6). The print symbol is a figure-eight lying on its side as embossed on Plate 1-5 above the I D box.*}

Example A-1. Let $A = (1, 3)$ and $B = (-2, 5)$. Find the distance s from A to B and the slope m of the line segment AB.

Solution

a. Find Δx and Δy :

$$\Delta x = -2 - 1 = -3$$
$$\Delta y = 5 - 3 = 2.$$

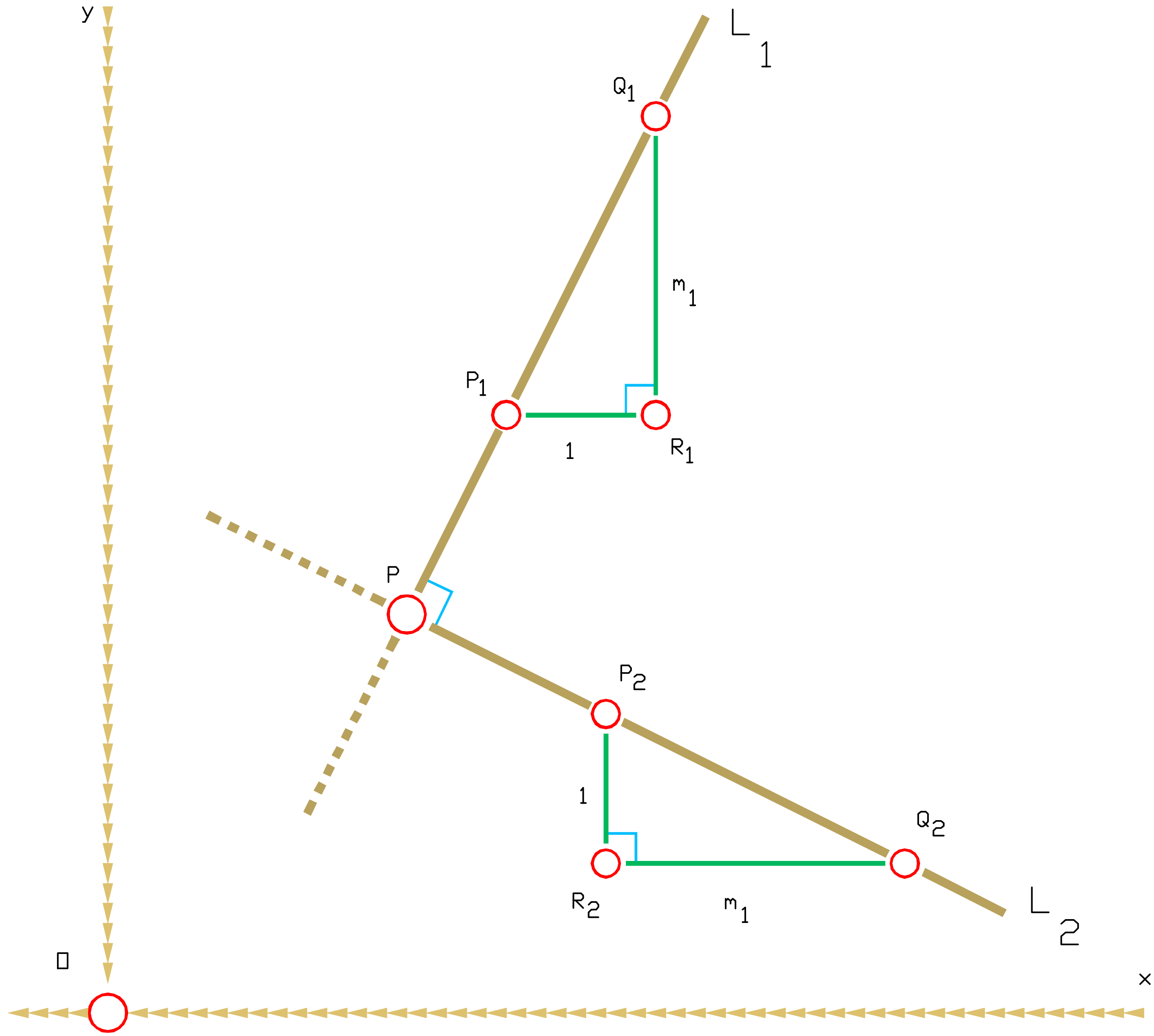


Fig. 1-5A

Figure 1-5A

b. Find s:

$$\begin{aligned} s &= \sqrt{\Delta x^2 + \Delta y^2} \\ &= \sqrt{(-3)^2 + 2^2} \\ &= \sqrt{9+4} \\ &= \sqrt{13}. \end{aligned}$$

c. Find m:

$$\begin{aligned} m &= \Delta y / \Delta x \\ &= -2/3. \end{aligned}$$

EXERCISES 1.3A

1. For each of the following pairs of points find the distance between the two and the slope of the line segment joining them.

a. $(1, 2)$ and $(5, 4)$	b. $(-1, -1)$ and $(2, 3)$
c. $(0, 5)$ and $(1, 2)$	d. $(1, 4)$ and $(1, -3)$
e. $(2, 3)$ and $(5, 3)$	f. $(1/2, 7)$ and $(1/2, 4)$

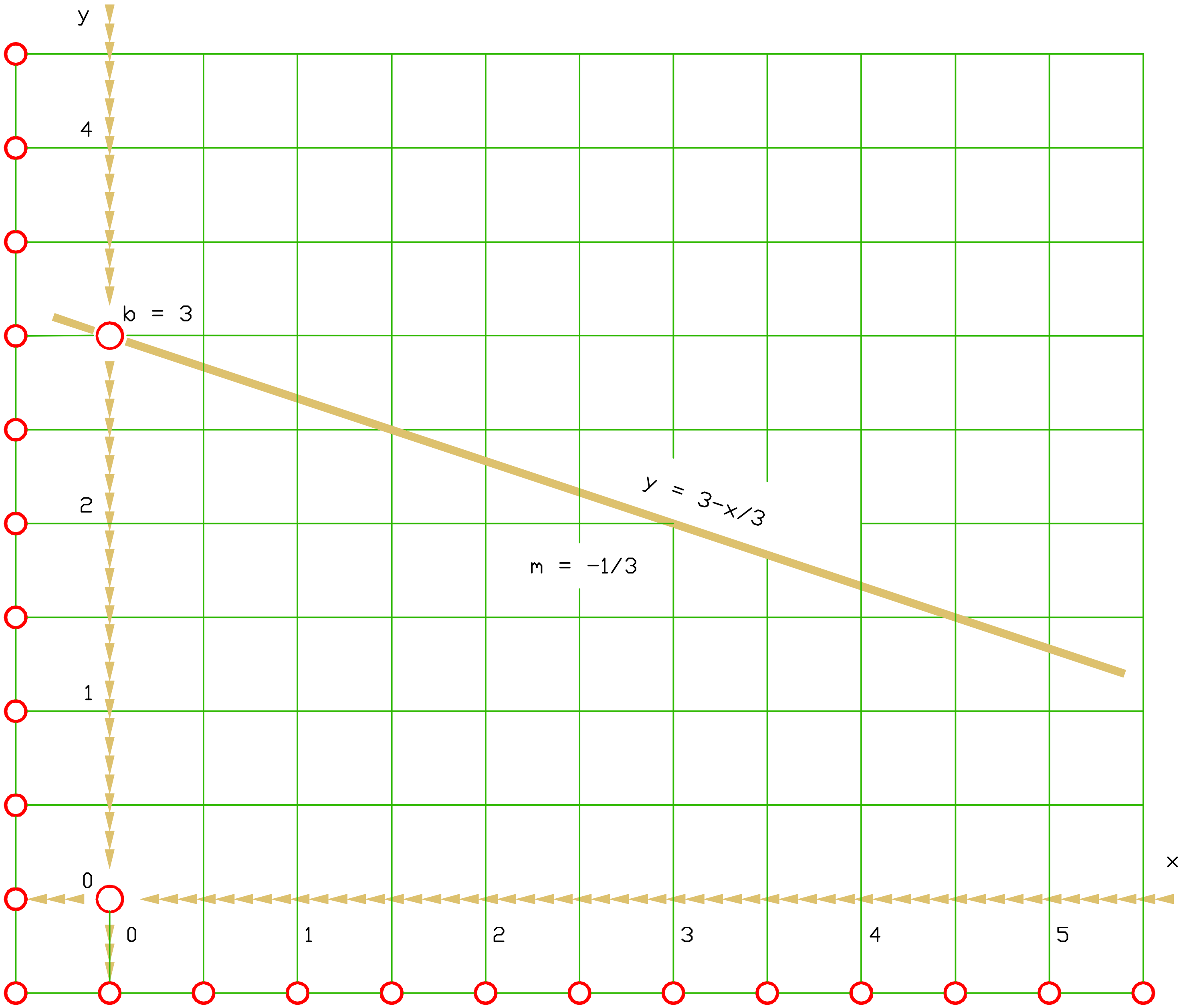
Perpendicular lines. {*Use Figure 1-5A.*} Figure 1-5A displays two perpendicular lines: L_1 with slope m_1 and L_2 with slope m_2. The point P is the intersection of L_1 and L_2. The figure shows the *rays* or *half-lines* extending toward the right from P. One of the lines must be rising and the other falling for, if both lines were rising, both rays would be pointing into the first quadrant and would form an acute angle, not a right angle. Similarly, the rays cannot both be falling. We have chosen L_1 as the rising line of the two. Let P_1 be any point on the rightward ray of L_1 from P. Let Q_1 be the point on the ray with a run or horizontal displacement, 1, from P and let R_1 be the point 1 unit horizontally to the right from P_1 vertically below Q_1. Since the ratio of rise to run is the slope of L_1, the rise from P_1 to Q_1 is m_1.

The line L_2 can be obtained by rotating L_1 clockwise around P through a right angle. Similarly, P_2, Q_2, R_2 are the respective images of P_1, Q_1, R_1 in the same rotation. Given that the displacement P_1R_1 points to 3 o'clock, the displacement P_2R_2 points to 6 o'clock. What are the directions of the displacements R_1Q_1 and R_2Q_2? Knowing the directions and lengths of these displacements, you can confirm that

$$m_2 = -1/m_1 .$$

The slope-intercept form. {*Use Plate 1-6.*} The most convenient form of the equation of a line with slope m is

$$y = mx + b,$$



KEYWORDS
slope
intercept
slope-intercept form

Fig. 1-6

Figure 1-6

the *slope-intercept* form. Here, b is the *y-intercept* of the line, that is, the ordinate of the point where the line crosses the y-axis. Figure 1-6 shows the graph of the line

$$x + 3y - 9 = 0.$$

Solving for y, we put the equation in the slope-intercept form

$$y = -\frac{x}{3} + 3.$$

From this, the y-intercept is 3 and the slope is $m = -1/3$, as we may readily verify by finding the run and rise for two conveniently chosen points, like $(0, 3)$ and $(3, 2)$.

All but vertical lines, which have no slope, can be put in the slope-intercept form. The slope-intercept form lends itself easily to graphical interpretation as in the following example.

Example B-1. {*Use Plate 1B-1.*} Sketch the graph of the straight line

$$2x + 3y = 6.$$

Solution
a. Put the equation in slope-intercept form:

$$y = -\frac{2}{3}x + 2.$$

b. Note the values of m and b:

$$m = -\frac{2}{3}, \qquad b = 2.$$

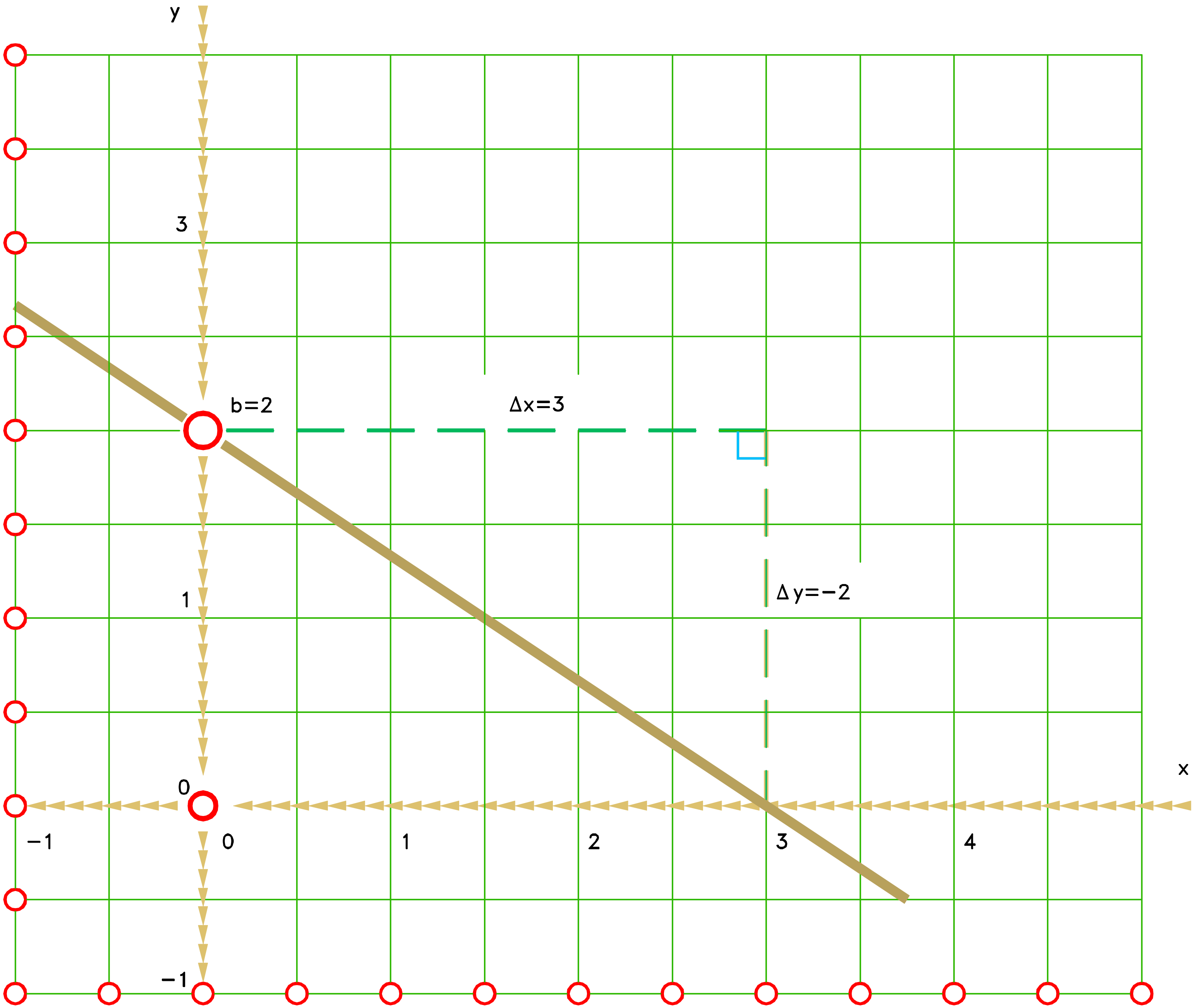
c. Find two points on the line and rule the line that connects them:

One point is the intersection $(0, b)$ of the line with the y-axis, namely, $(0, 2)$. A second point can be found conveniently by taking a run of 3 and to obtain the corresponding rise -2 from the first point to get $(3, 0)$. See Fig. 1B-1.

EXERCISES 1.3B

1. Graph the following equations by sketching the slope and y-intercept.

a.	$y = 5x + 2$
b.	$2x + 3y = 6$
c.	$x - 2y = 5$



EXAMPLE 1B-1 ◇

run ◇

rise ◇

slope ◇

intercept ◇

Fig. 1B-1

Figure 1B-1

1.4 The Point-slope Form.

{*Use Plate 1-7.*}

A straight line is determined by its slope and any one of its points. For example, Figure 1-7 shows the line with slope 3/2 passing through the point $A = (1, -2)$. To find the equation of the line, let $X = (x, y)$ represent any point on the line and calculate the run and rise from A to X.

$$\Delta x = x - 1 \quad \text{and} \quad \Delta y = y + 2$$

and from

$$m = \Delta y \,/ \Delta x$$

obtain

$$\frac{3}{2} = \frac{y+2}{x-1}.$$

This equation can be written in the general form

$$3x - 2y = 7.$$

Compare Example B-2 below.

In general, if $A = (a, b)$ is a point on a line, L, with slope m, the point-slope form of the equation of L is

$$\frac{y-b}{x-a} = m.$$

Example B-2. {*Use Plate 1B-2.*} What are the equations of the lines drawn in Parts a and b of Fig. 1B-2?

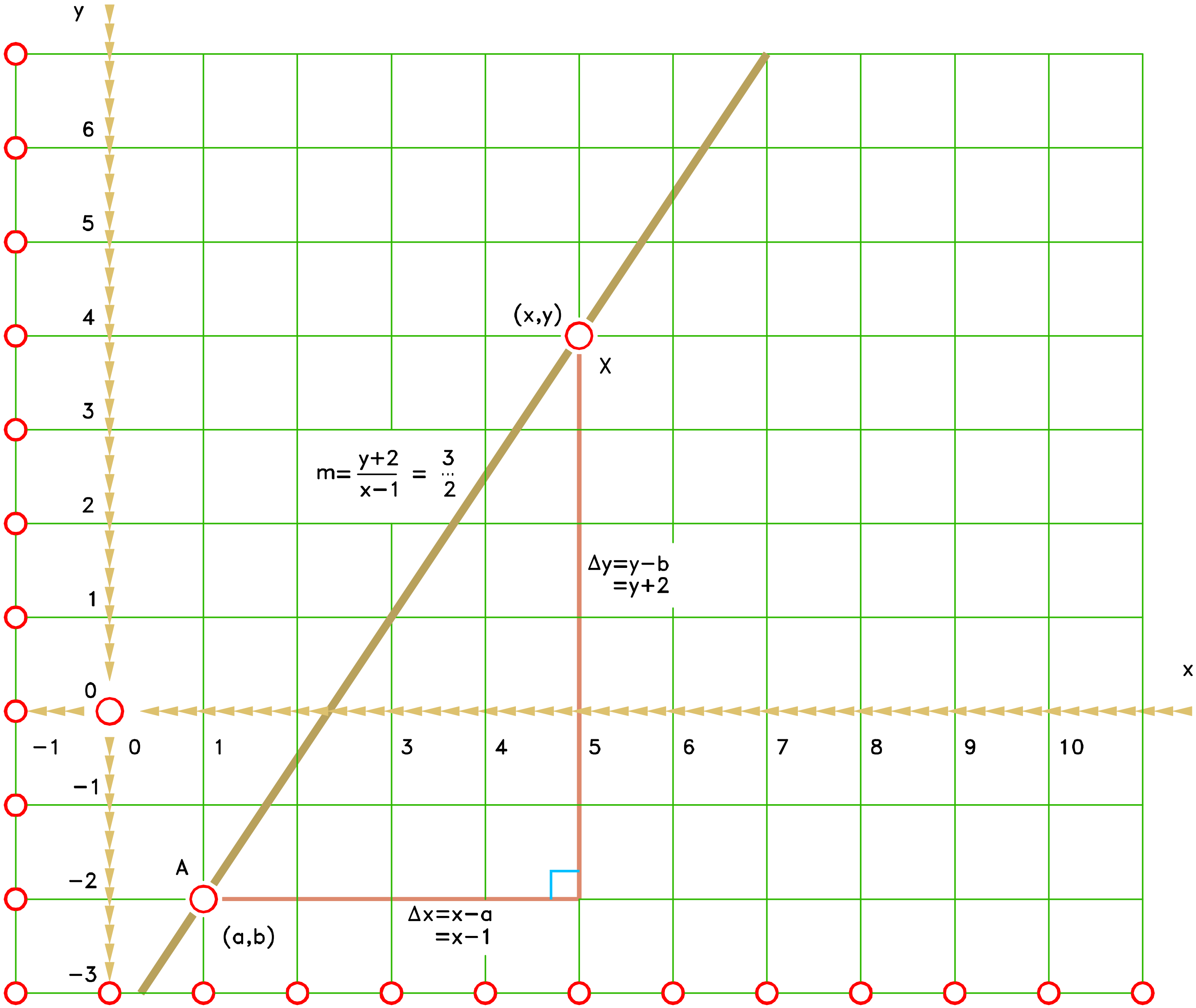
{*Graphics note: The two parts of this graphic are* sketches. *In a sketch, only partial information is given and the rest of the information must be inferred. Part a presents a line connecting points A and B without specified coordinates. The axes and origin are shown and labeled. Gridlines are shown but the difference in coordinates between gridlines has to be inferred from the points* $(5, 0)$ *and* $(0, 5)$ *displayed as the solid buttons on the* x*- and* y*-axes. Only the coordinate buttons* $x = 0$, $x = 5$, $y = 0$ *and* $y = 5$ *give voice messages when pressed.*

In Part b, the displayed line connects points P_1 *and* P. *On the axes, there are buttons at* $(1, 0)$ *and* $(0, 5)$. *The voiced coordinate buttons are* $x = 0$, $x = 1$, $y = 0$, $y = 5$. *Both sketches use unit spacing for the gridlines but that may vary in the future.*}

Solution

Part a.

a. Pick any two points on the line:



KEYWORDS
- Straight line
- slope
- point−slope form

Fig. 1−7

Figure 1−7

In this example, a simple choice is to take the two points where the line intersects the axes,

$$(x_1, y_1) = (0, 2), \qquad (x_2, y_2) = (4, 0).$$

b. Use the point-slope form: Calculate the ratio of rise to run for these points to get the slope,

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 2}{4 - 0} = -\frac{1}{2}.$$

Then, insert the coordinates of either point, say $a = 4$ and $b = 0$, in the point-slope form to get

$$\frac{y - 0}{x - 4} = -\frac{1}{2}$$

or

$$x + 2y - 4 = 0.$$

Part b.
In this part, we use the point-slope form directly:
a. Pick a point:
Here we pick

$$P_1 = (2, 3).$$

b. Choose a run Δx from P_1 and find the corresponding rise: We choose the run

$$\Delta x = 2$$

from P_1 to $x = 4$ and find a drop of 4 of units to the line or the rise,

$$\Delta y = -4.$$

c. Calculate the slope:

$$m = \frac{\Delta y}{\Delta x} = -\frac{4}{2} = -2.$$

d. Write the point-slope form:

$$\frac{y - 3}{x - 2} = -2$$

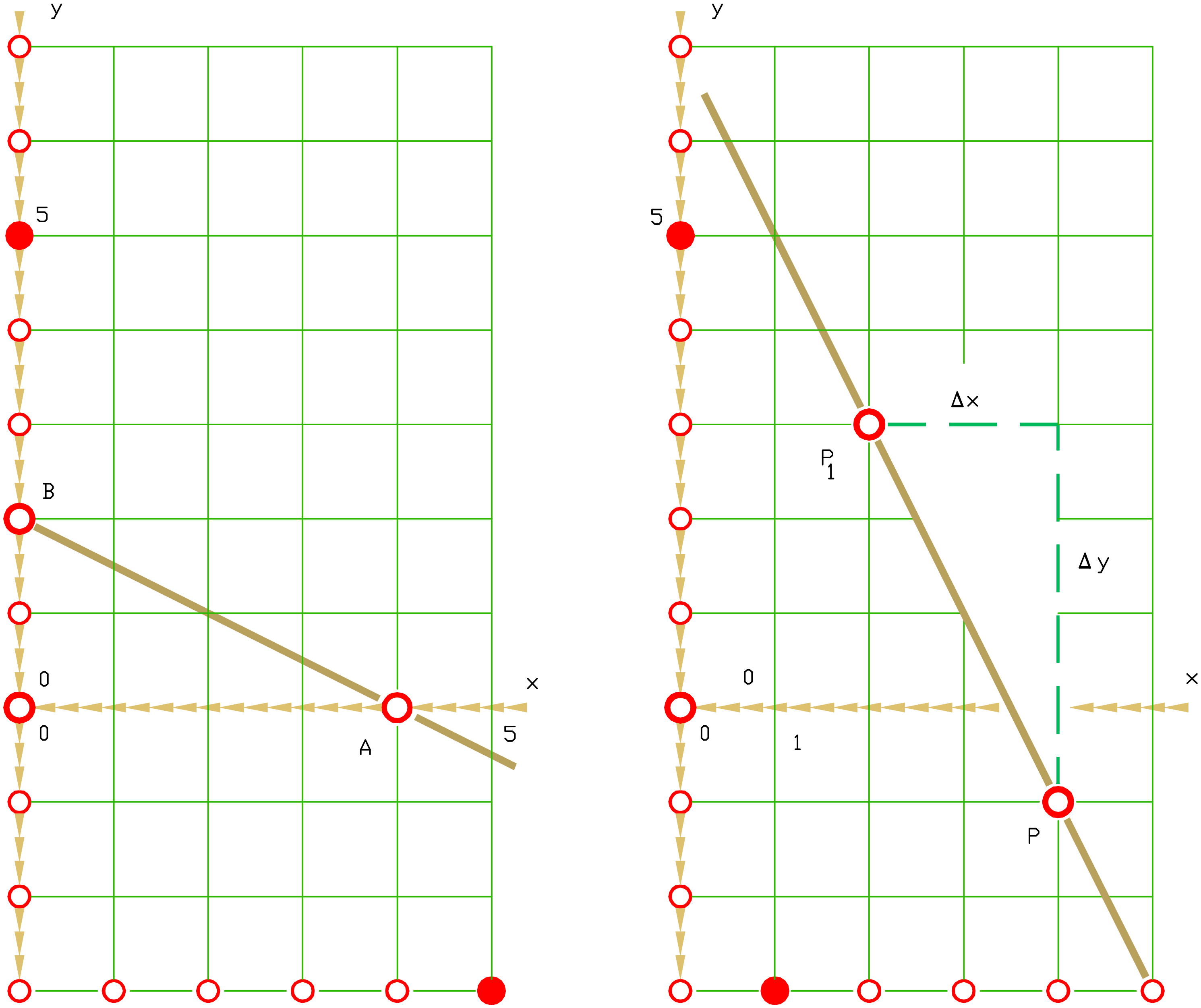
or, in slope-intercept form,

$$y = -2x + 7.$$

EXERCISES 1.4.

1. Find the equation of the straight line through the given point with the given slope. Sketch.

	point	slope
a.	$(1, 1)$	1
b.	$(3, 2)$	-1
c.	$(4, 1)$	-2
d.	$(3, 2)$	0



EXAMPLES 1B-2 ◇

line ◇

y-intercept ◇

x-intercept ◇

run ◇

rise ◇

slope ◇

sketch ◇

Fig. 1B-2

Figure 1B-2

1.5 The Point-point Form.

{*Use Plate 1-8.*}

Since a line is determined by any two of its points, we should be able to find its equation in terms of the coordinates of those points. Fig. 1-8 shows the line through the points $P_0 = (1, 1)$ and $P_1 = (3, 2)$. To find the equation of this straight line, we calculate the run and the rise from P_0 to P_1:

$$\Delta x = 3 - 1 = 2 \quad \text{and} \quad \Delta y = 2 - 1 = 1.$$

From this we get the slope,

$$m = \frac{\Delta y}{\Delta x} = \frac{1}{2}.$$

Inserting $P_0 = (1, 1)$ in the point-slope form, we get

$$\frac{y-1}{x-1} = \frac{1}{2}$$

or

$$x - 2y = -1.$$

In general, if $P_0 = (a_0, b_0)$ and $P_1 = (a_1, b_1)$, the point-point equation of the line has the form

$$\frac{y-b_0}{x-a_0} = \frac{b_1-b_0}{a_1-a_0}.$$

Example C-1. Find the equation of the straight line through the point $P_1 = (2, 3)$ with slope $m = -2$.

Solution

a. First sketch the line as in Plate 1C-1.

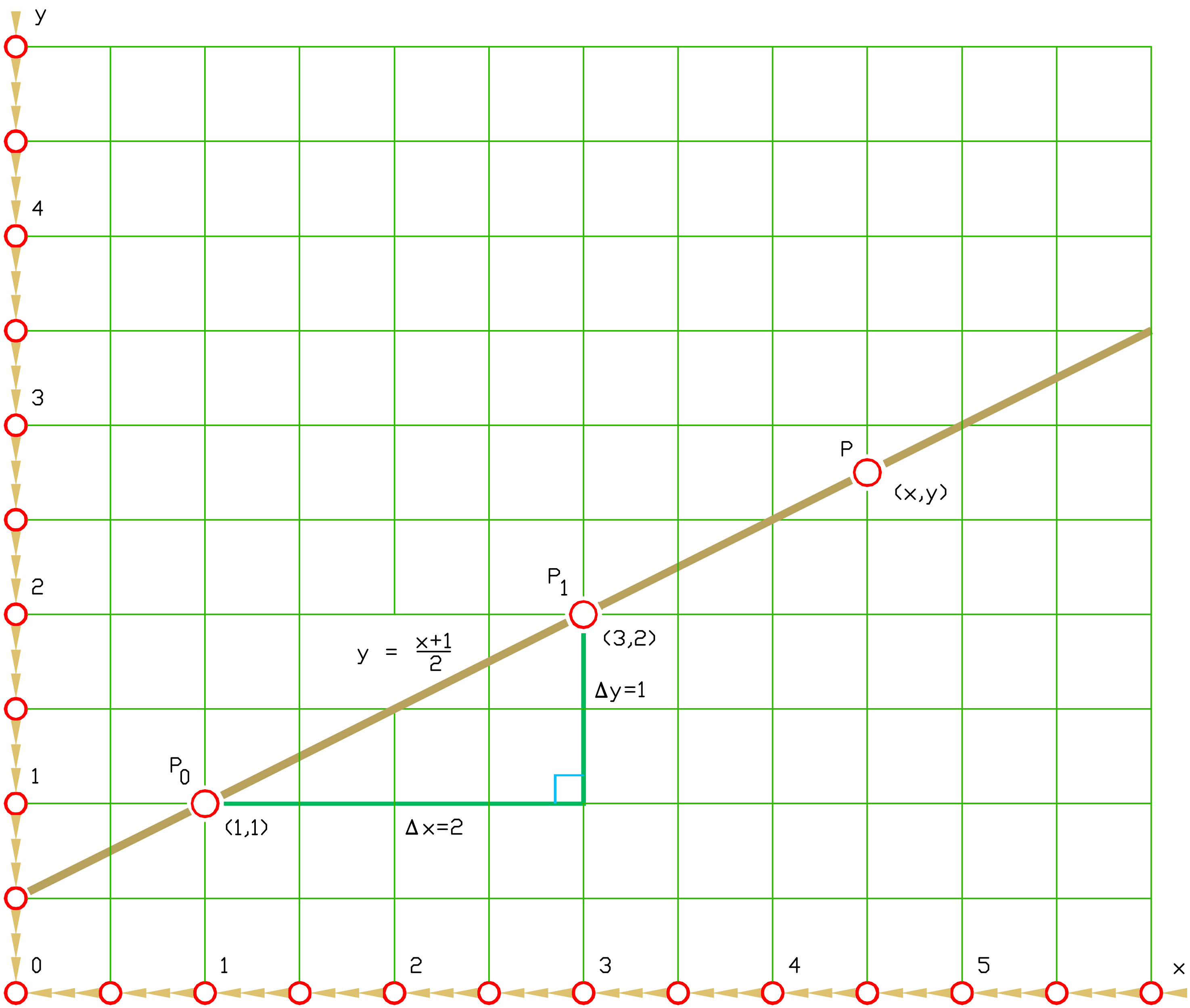
{*Graphics note: This figure is even sketchier than before. The grid lines are omitted. Now there are only tick marks extending from the coordinate buttons to the axes. Only the solid coordinate buttons, $x = 0$, $x = 1$, $y = 0$, $y = 5$, are voiced. The jump in x between successive buttons is one half; in y, it is 1.*}

In Fig. 1C-1, first mark the point P_1. Then, consistent with the slope, -2, take a horizontal displacement of 2 units from P_1 followed by a downward vertical displacement of 4 units to a point P on the line.

b. Write the equation in point-slope form:

$$\frac{\Delta y}{\Delta x} = m,$$

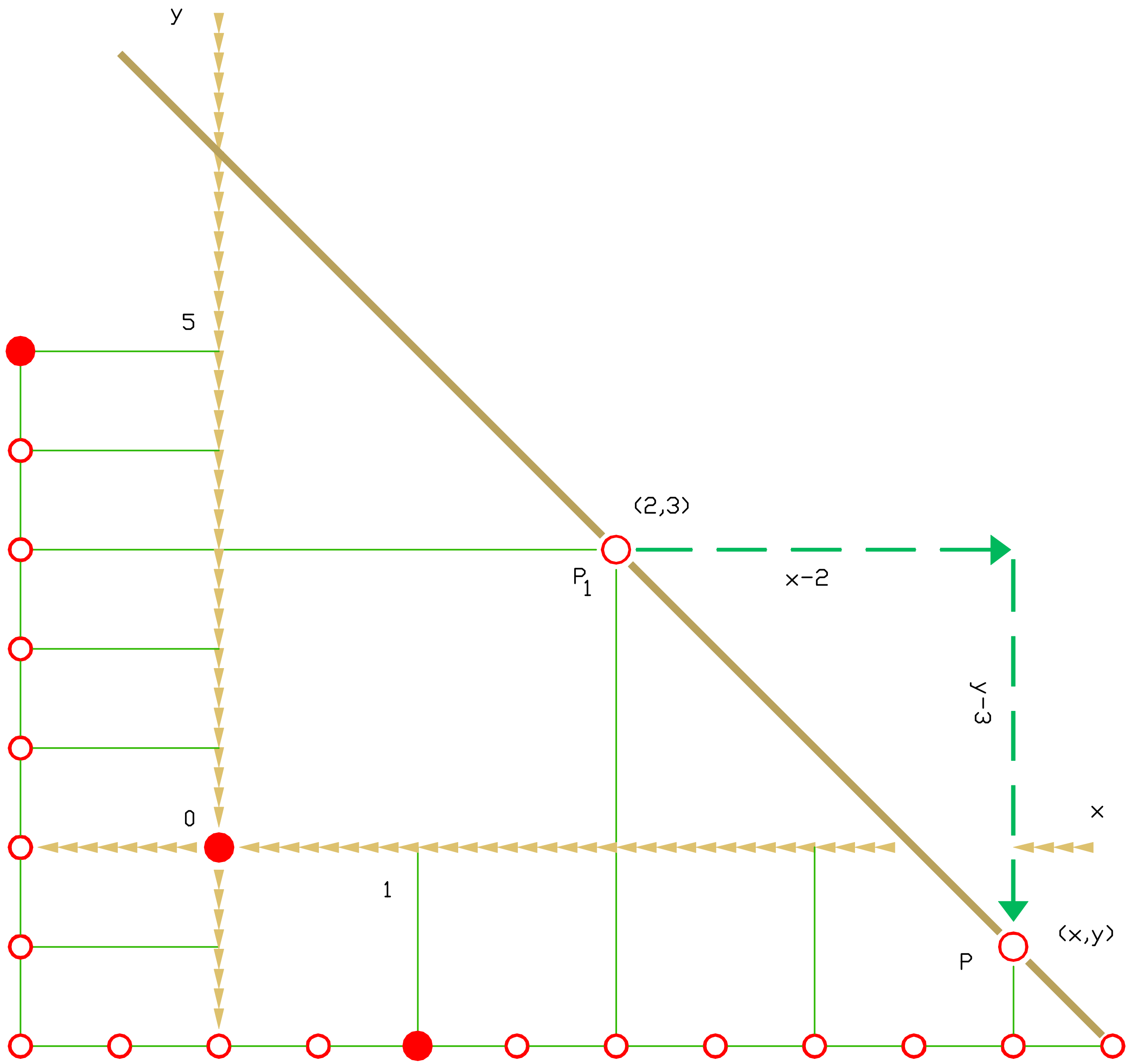
$$\frac{y-3}{x-2} = -2,$$



KEYWORDS
straight line
slope
point-point form

Fig. 1-8

Figure 1-8



Example 1C-1 ◇

slope ◇

sketch ◇

tick mark ◇

Figure 1C-1

Figure 1C-1

Example 1C-2

point-point form

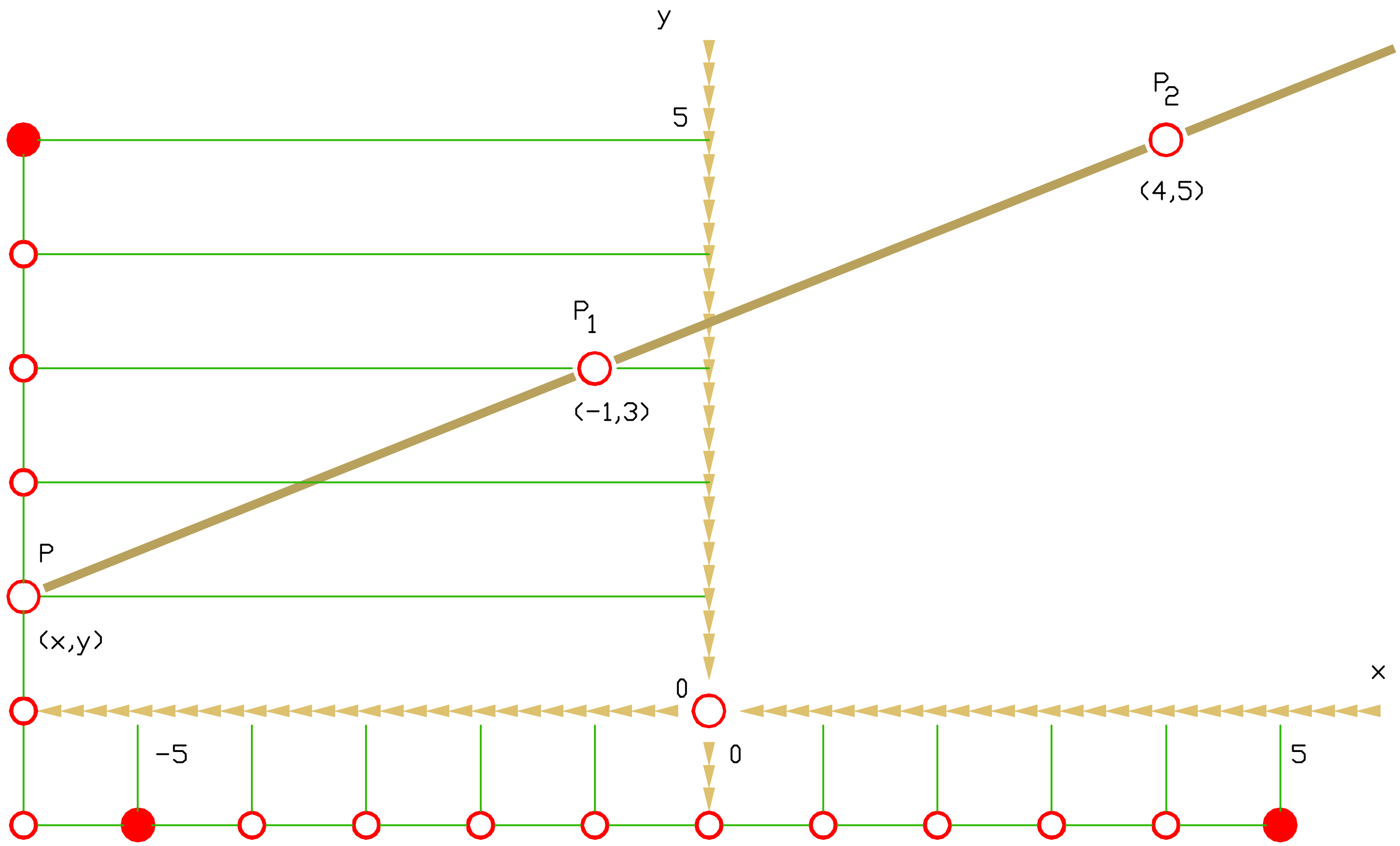


Fig. 1C-2

Figure 1C-2

$$y = -2x + 7$$

Example C-2. Find the equation of the straight line through the points $(-1, 3)$ and $(4, 5)$. {*See Fig. 1C-2*}.

Solution

Use the point-point form: For an arbitrary point $P = (x, y)$ on the line,

$$m = \frac{y-3}{x+1} = \frac{5-3}{4+1} = \frac{2}{5},$$

hence,

$$5y - 15 = 2x + 2,$$

$$2x - 5y + 17 = 0.$$

EXERCISES 1.5

1. Find the equation of the straight line through each of the pairs of points,

a.	$(1, 2)$ and $(5, 4)$	b.	$(-1, -1)$ and $(2, 3)$
c.	$(0, 5)$ and $(1, 2)$	d.	$(1, 4)$ and $(1, -3)$
e.	$(-2, 3)$ and $(5, 3)$	f.	$(2, 2)$ and $(5, 5)$

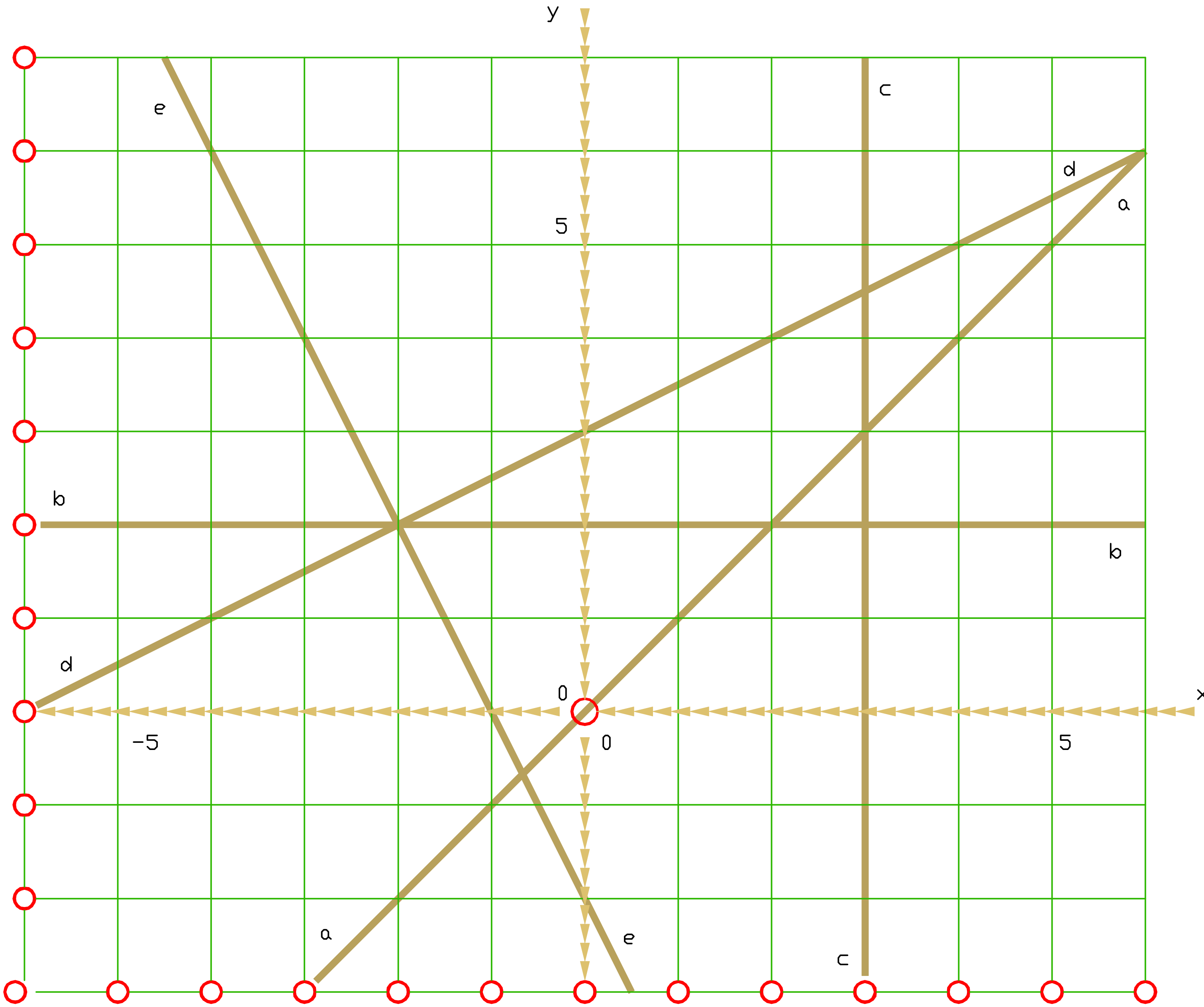
2. {*Use Plate 1XB2.*} Fig. 1XB2 displays lines a, b, c, d, e. What are their (approximate) equations?

1.6 Intersecting and Parallel Lines.

To find the coordinates of the point where two lines intersect, find the simultaneous solution of the equations of the lines. Compare Example D-1. You might like to practice your algebra skills and prove that lines

$$y_1 = m_1 x + b_1 \text{ and } y_2 = m_2 x + b_2$$

meet in precisely one point if and only if m_1 and m_2 are different and to obtain a formula for that point.



Exercise 1-B2

line

sketch

equation

Fig. 1-Ex. B2

Figure 1-X B2

Parallel lines. {*Use Plate 1-6A.*} Since parallel lines are defined in elementary geometry as lines that do not meet and lines that have different slopes must meet, it follows that parallel lines have the same slope. Moreover, if the lines $y = mx + b_1$ and $y = mx + b_2$ having the same slope do meet, then $b_1 = b_2$ so that the equations describe the same line. So, if lines with the same slope are distinct, they are parallel. In coordinate geometry, we extend the definition of *parallel* slightly to mean only that parallel lines have the same direction or slope. In the extended definition, a line is parallel to itself.

Because parallel lines have the same slope, the ratio of rise to run for two points P and Q on a line is equal to the ratio for any two points P' and Q' on a parallel (See Figure 1-6A). If we draw the horizontal and vertical displacements from the first to the second point of each pair, we form similar right triangles, $\triangle PRQ$ and $\triangle P'R'Q'$. This last property could be taken as the geometrical definition of parallel.

Before you attempt to solve two linear equations simultaneously, it is a good idea to check that the slopes of the lines are not equal to be sure that the lines are not parallel,

Note. In the foregoing discussion of intersecting and parallel lines, we tacitly assumed that the lines have slopes. If some of the lines don't have slopes, that is, if any of the lines are vertical, we have a simple special case which we leave for you to deal with.

Example D-1. {*Use Plate 1D-1.*} Are the lines $2x+2y = 5$ and $y = 3x+1$ parallel? If not, find their intersection. Sketch.

Solution

a. Find the slopes m_1 and m_2 of the respective lines:

$$\begin{array}{lll} \text{For } 2x + 2y = 5, & y = -x + 5/2; & m_1 = -1. \\ & y = 3x + 1; & m_2 = 3. \end{array}$$

Since $m_1 \neq m_2$, the lines are not parallel.

b. Sketch:

The lines are sketched in Fig. 1D-1.

{*Graphics note: The plate, Fig. 1D-1, displays the coordinate grid so that you can estimate the coordinates of the intersection point by touch. Observe that the jump in coordinates between successive grid lines is one half unit.*}

c. Solve the simultaneous equations:

The algebraic operations in the solution are indicated by the labels.

$$\begin{array}{rrcrcrr} \text{[A]} & 2x & + & 2y & = & & 5 \\ \text{[B]} & 3x & - & 1y & = & - & 1 \\ \text{[2B]} & 6x & - & 2y & = & - & 2 \\ \text{[A+2B]} & 8x & & & = & & 3. \end{array}$$

KEYWORDS ◇

parallel lines ◇

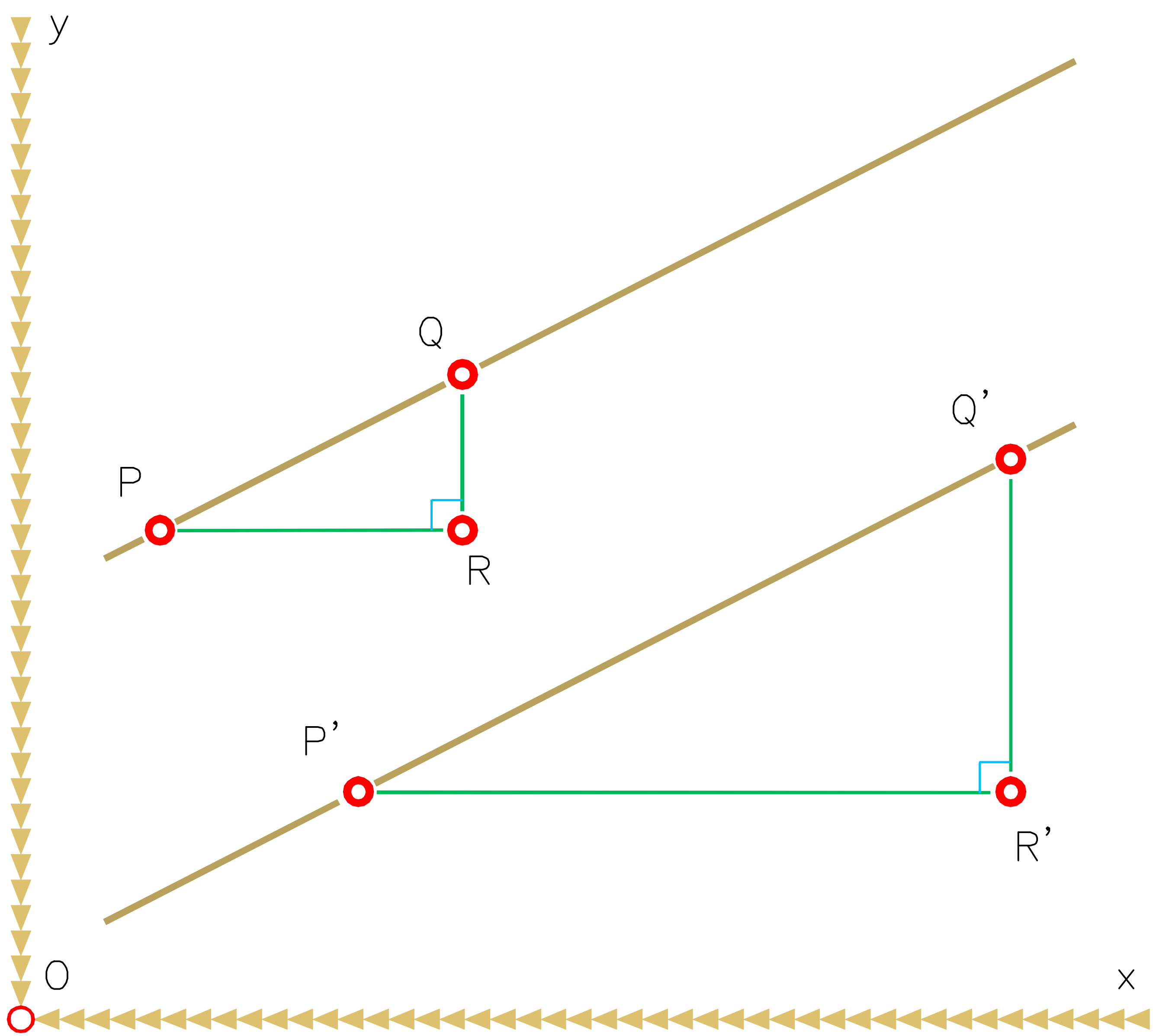


Fig. 1–6a

Figure 1-6A

Example 1D-1

intersection

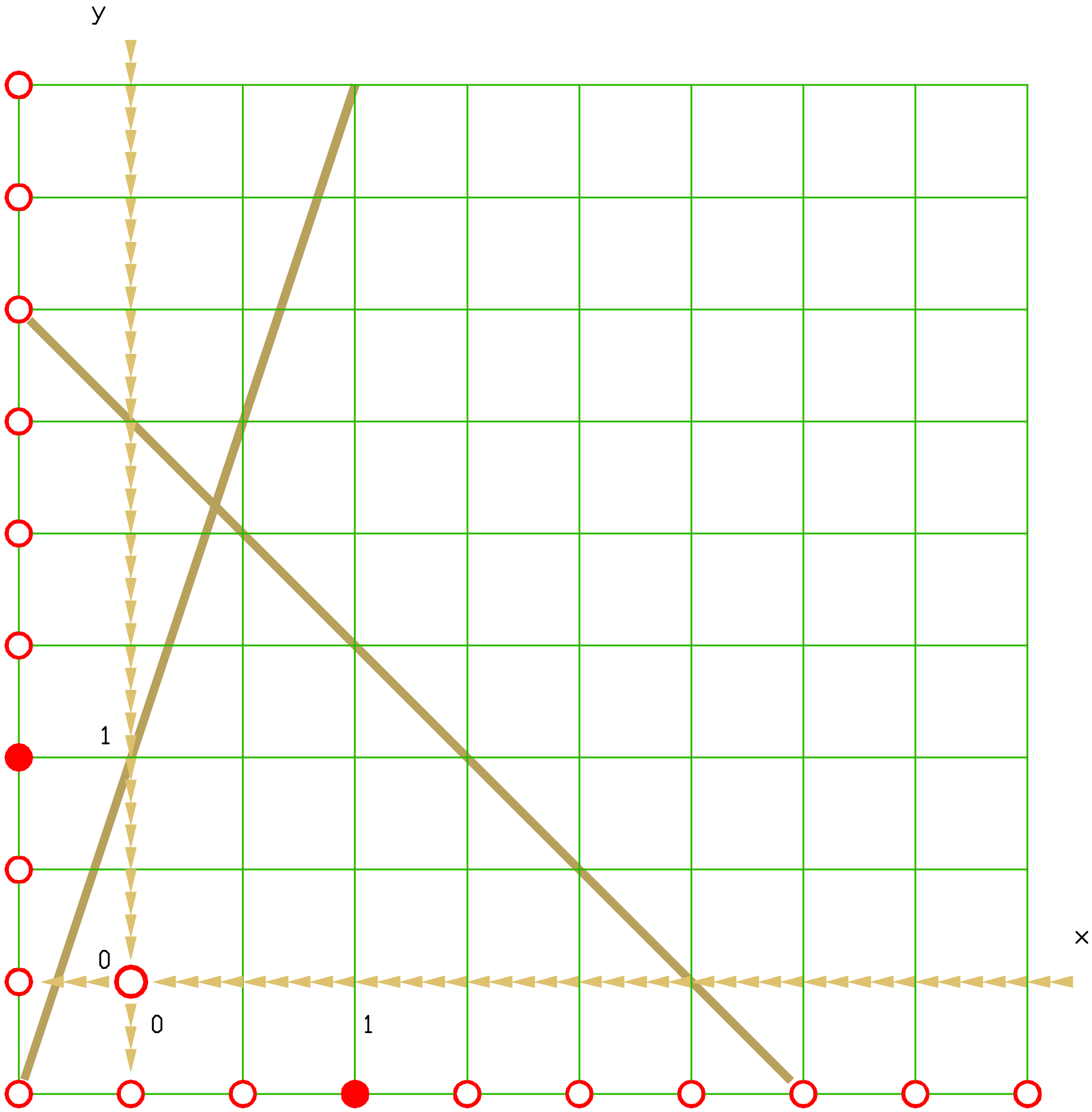


Fig. 1D-1

Figure 1D-1

Example 1E-1 ◇

midpoint ◇

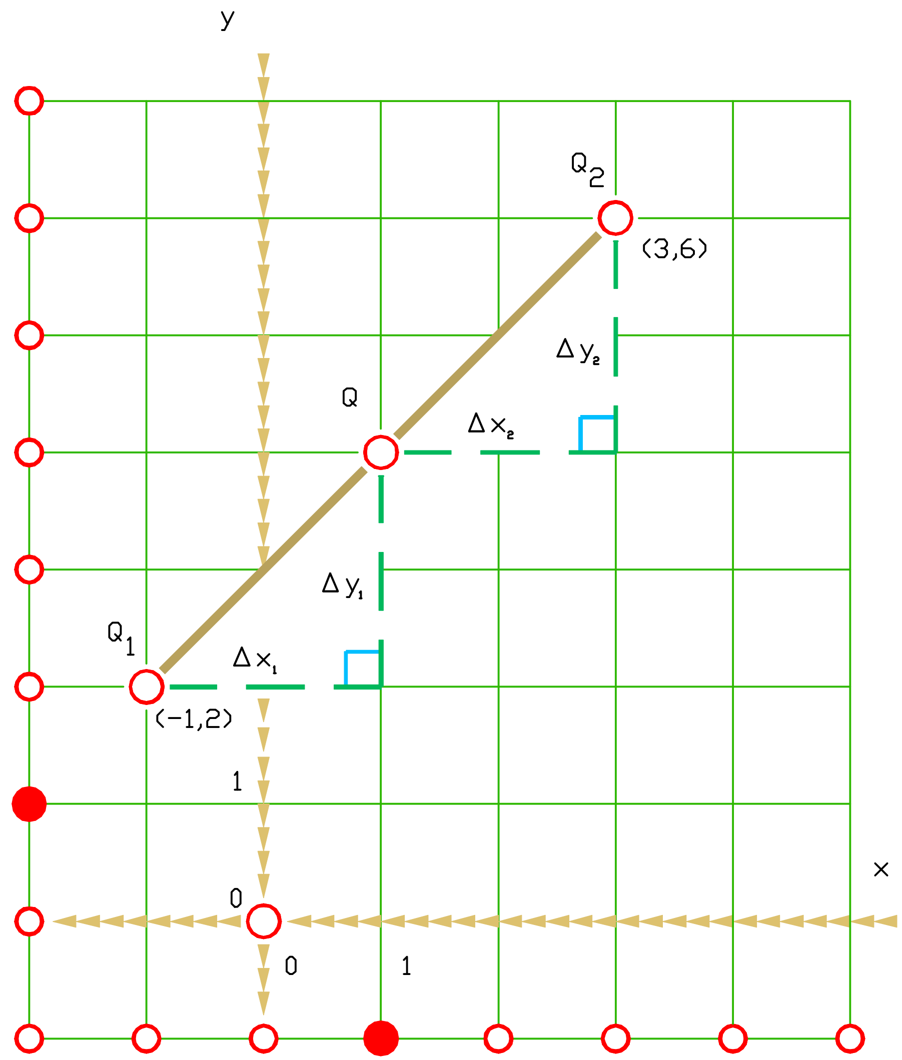


Fig. 1E-1

Figure 1E-1

Hence,

$$x = 3/8.$$

Enter this value in $y = 3x + 1$ to get

$$y = 17/8 = 2\ {}^{1}/_{8}.$$

The point of intersection is

$$P = (3/8,\ 17/8).$$

EXERCISES 1.6.

1. Determine whether each pair of lines is parallel and, if not, find their intersection:

a. $x + y = 2; \quad x - y = 1$
b. $x - 2y = 1; \quad 2x = 4y + 3$
c. $x = 2; \quad y = 3$
d. $x = 1; \quad y = x + 1.$

1.7 Geometry Problems.

Having mastered skills A through D, you are prepared to solve more complicated problems by combining these skills as in the following examples:

Example E-1. {*Use Plate 1E-1.*} Find the midpoint of the line segment with endpoints $Q_1 = (-1, 2)$ and $Q_2 = (3, 6)$.

Solution.
a. Sketch:
Figure 1E-1 shows the midpoint $Q = (x, y)$ of the segment Q_1Q_2, the run Δx_1 and rise Δy_1 from Q_1 to Q, and the run Δx_2 and rise Δy_2 from Q to Q_2.
b. Find the coordinates of Q:
Since the lengths of the segments Q_1Q and QQ_2 are equal, the two right triangles in the figure are congruent. Hence

$$\Delta x_1 = \Delta x_2 \quad \text{and} \quad \Delta y_1 = \Delta y_2.$$

Consequently, x and y are the averages of the respective cordinates of Q_1 and Q_2, that is,

$$x = \frac{-1 + 3}{2} = 1 \quad \text{and} \quad y = \frac{2 + 6}{2} = 4$$

or

$$Q = (1, 4).$$

Example 1E-2

perpendicular bisector

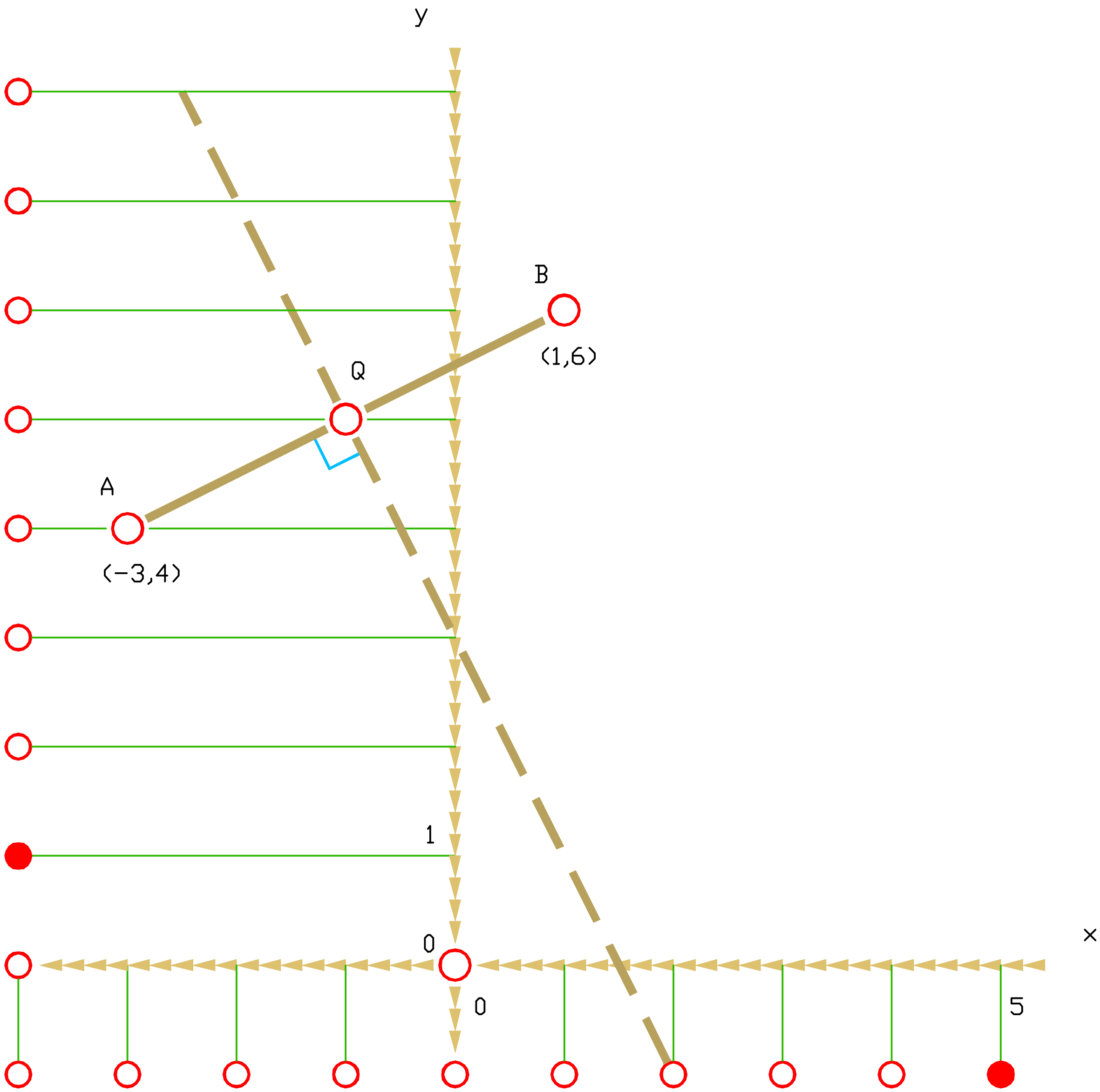


Fig. 1E-2

Figure 1E-2

Example E-2. {*Use Plate 1E-2.*} Find the equation of the perpendicular bisector of the line segment AB, where $A = (-3, 4)$ and $B = (1, 6)$. The *perpendicular bisector* of a segment is the line perpendicular to the segment that passes through its midpoint.
Sketch.

Solution.
a. Sketch:
The sketch in Figure 1E-2 shows the midpoint Q of the segment AB and displays the perpendicular bisector as a dashed line.
b. Find the slope of AB:

$$m_{AB} = \frac{6-4}{1+3} = \frac{1}{2}.$$

c. Slope of the perpendicular bisector:
From Section 1.3,

$$m = -1/m_{AB} = -2.$$

d. Midpoint of AB:
Its coordinates are the averages of those of A and B, namely,

$$Q = (-1, 5).$$

e. Equation of the perpendicular bisector in point-slope form:

$$m = -2 = \frac{y-5}{x+1}$$

or

$$y = -2x + 3.$$

Example E-3. {*Use Plate 1E-3.*} Find the coordinates of the foot P of the perpendicular from the point $A = (2, 3)$ to the line $x + y = 3$. The *foot* of the perpendicular is the point where it meets the line.

Solution.
a. Sketch:
The sketch, Figure 1E-3, uses a scale of one half unit between successive coordinate buttons.
b. Slope of the line:
Put $x + y = 3$ in the point-slope form, $y = -x + 3$, to get

$$m = -1.$$

c. Slope of the perpendicular:

$$m_{\perp} = -1/m = 1.$$

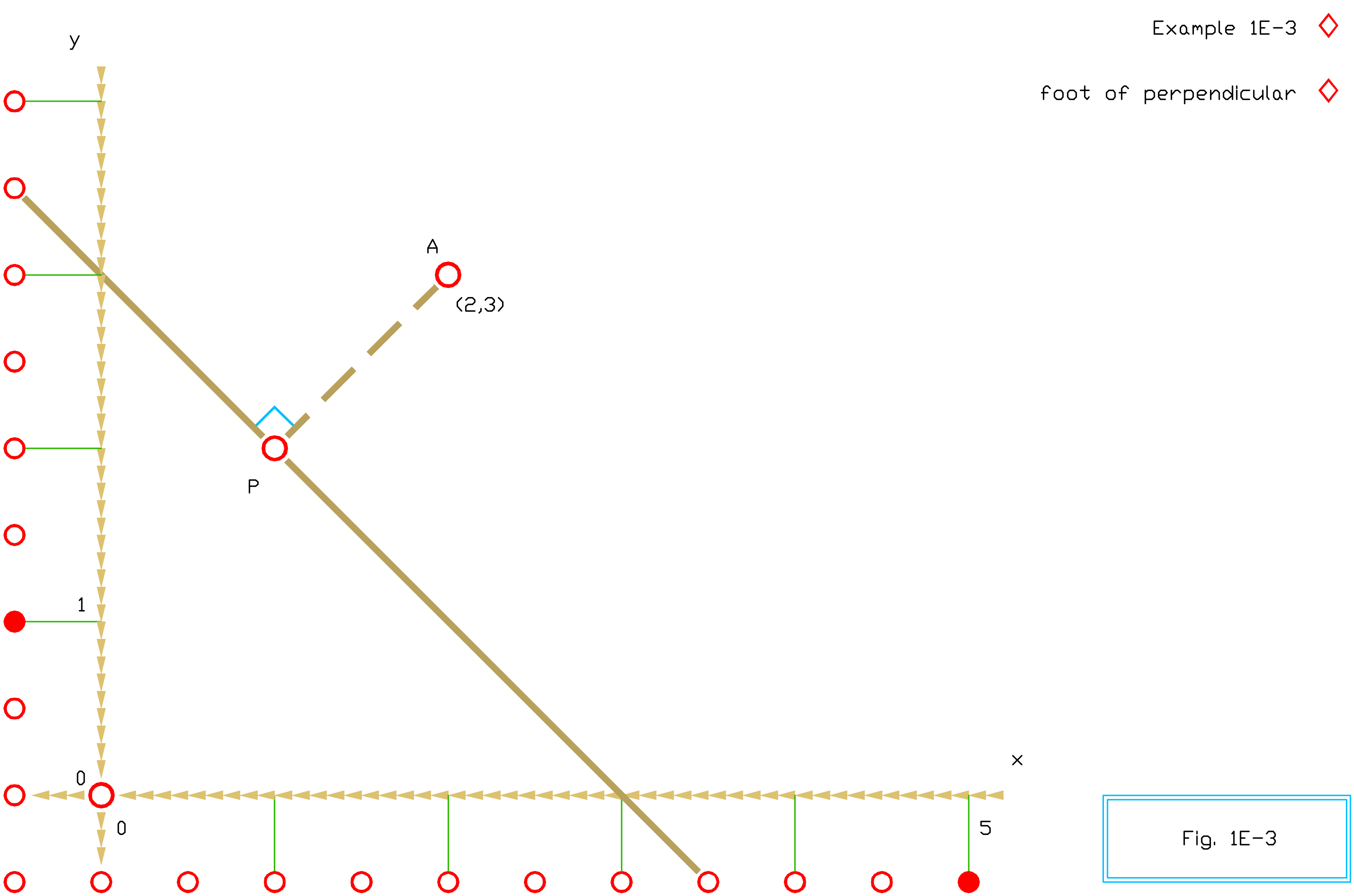


Figure 1E-3

d. Equation of the perpendicular line through $(2, 3)$:

$$m_\perp = 1 = \frac{y-3}{x-2}.$$

e. Foot of the perpendicular:
This is the intersection of the perpendicular lines found by solving the simultaneous equations.

$$\begin{array}{rrcr} [\mathrm{A}] & y & = & -x+3 \\ [\mathrm{B}] & y & = & x+1 \\ [\mathrm{A{+}B}] & 2y & = & 4 \end{array}$$

Hence,

$$y = 2 \qquad \text{and} \qquad x = y - 1 = 1.$$

$$P = (1, 2).$$

Example E-4. {*Use Plate 1E-4.*} The foot of the perpendicular from the origin O to the line L is the point $N = (-1, 3)$. Find the equation of L. Sketch.
Remark. The segment $\overline{ON}$ is called the *principal normal* or simply the *normal* to L.

Solution.
a. Sketch:
In Figure 1E-4, the normal is shown as a dashed line. Successive coordinate buttons are one half unit apart.
b. Slope of the perpendicular segment, $\overline{ON}$:

$$m_\perp = \frac{3-0}{-1-0} = -3.$$

c. Slope of L:

$$m_L = -1/m_\perp = 1/3$$

d. Equation of L: The point-slope form is

$$m_L = \frac{1}{3} = \frac{y-3}{x+1}.$$

Hence,

$$x - 3y + 10 = 0.$$

EXERCISES 1.7

1. The *distance* from a point to a line is the length of the perpendicular from the point to the line. What is the distance from $P = (2, 3)$ to the line $L:\ y + x + 1 = 0$? Sketch.

Example 1E-4

principal normal

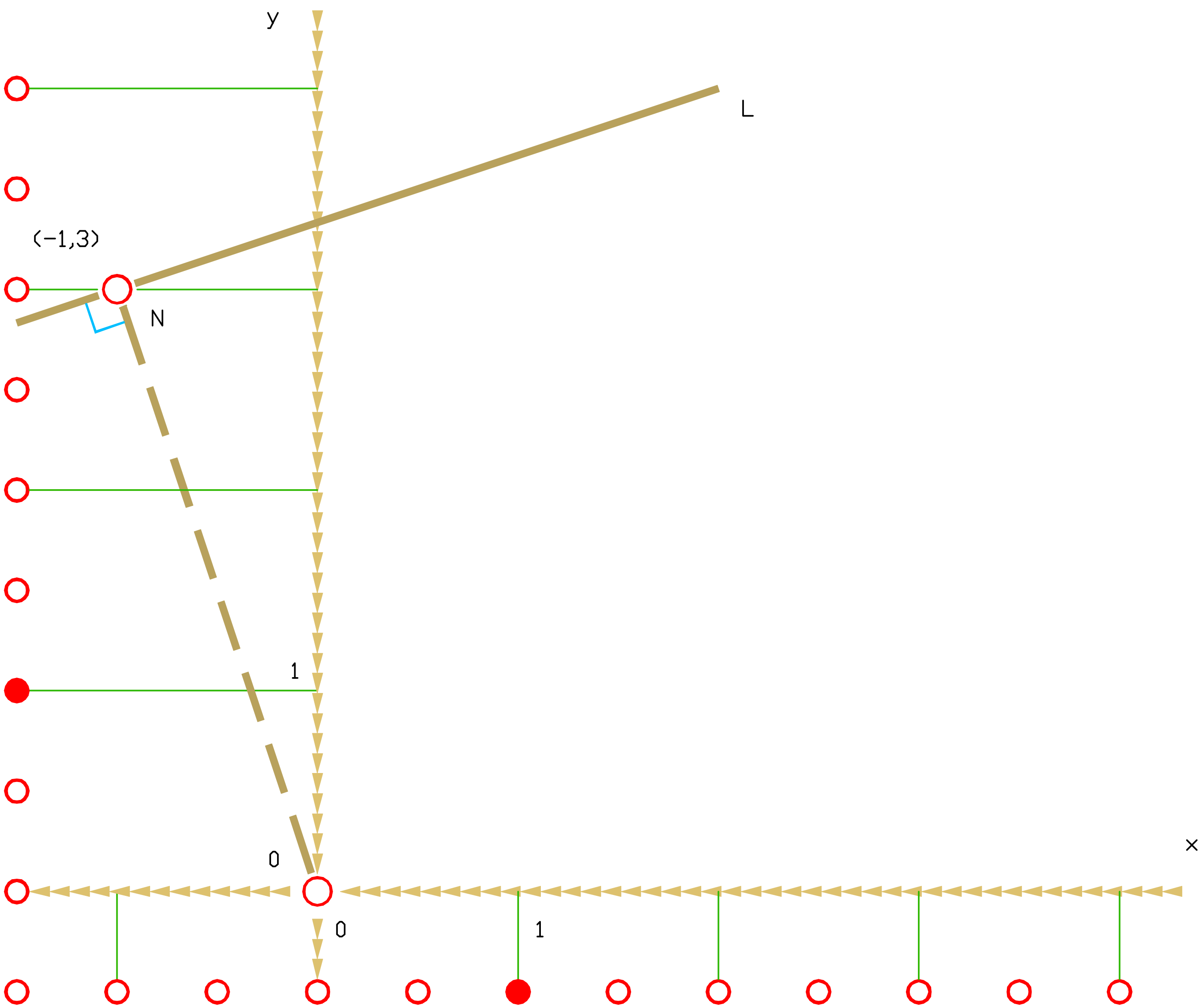


Fig. 1E-4

Figure 1E-4

2. What is the equation of the perpendicular bisector of the line segment AB, where $A = (0, 1)$ and $B = (-4, -3)$. Sketch.

3. Find the equation of the straight line which passes through the point $(1, 2)$ and is parallel to the line $2x + 4y = 7$.

4. Find the coordinates of the foot of the perpendicular from the point $(0, 1)$ to the line $x + y = 7$.

1.8 Circles {*Use Plate 1-9*}

A circle is the locus of all points at a given distance, the *radius*, from a given point, the *center*. The distance formula from subsection 1.2 immediately yields the equation of a circle. For example, the circle of radius 2 centered at the point $(3, 2)$, shown in Fig. 1-9, the set of points whose distance from $(3, 2)$ is 2. The distance formula yields the equation of this circle in the form

$$\sqrt{(x-3)^2 + (y-2)^2} = 2$$

or

$$(x-3)^2 + (y-2)^2 = 4$$

or, if you wish,

$$x^2 + y^2 - 6x - 4y + 9 = 0.$$

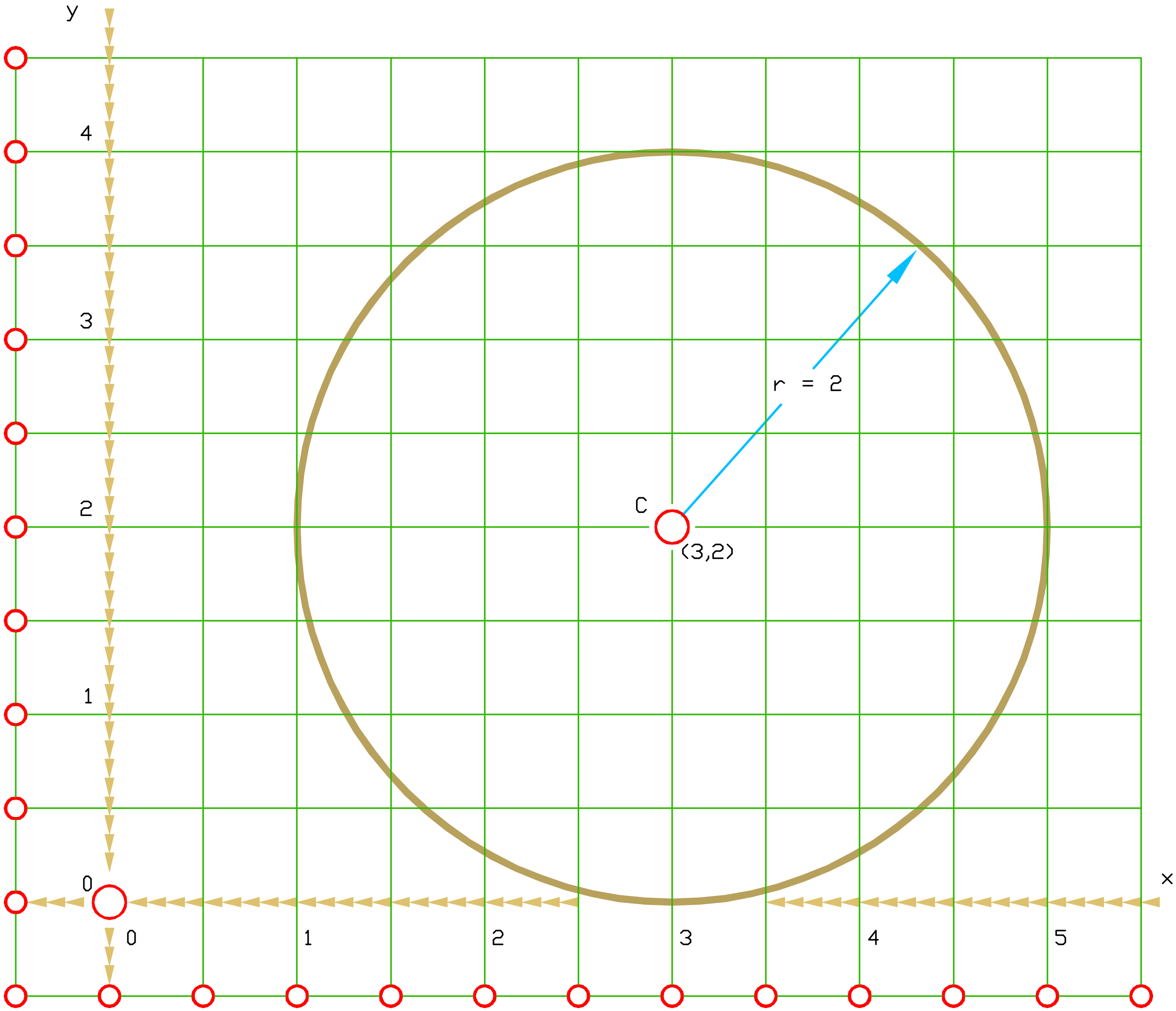
Example F-1. (*Use Plate 1F-1.*) Find the equation of the circle with radius $r = 3$ and center $C = (1, 4)$.

Solution.
a. Sketch:
See Fig. 1F-1.
b. Let $P = (x, y)$ represent any point on the circle and set $\Delta x = x - 1$, $\Delta y = y - 4$. Since r is the distance from P to C,

$$(\Delta x)^2 + (\Delta y)^2 = r^2$$

$$(x-1)^2 + (y-4)^2 = 9$$

$$x^2 + y^2 - 2x - 8y + 8 = 0$$



KEYWORDS
circle
center
distance
radius

Fig. 1-9

Figure 1-9

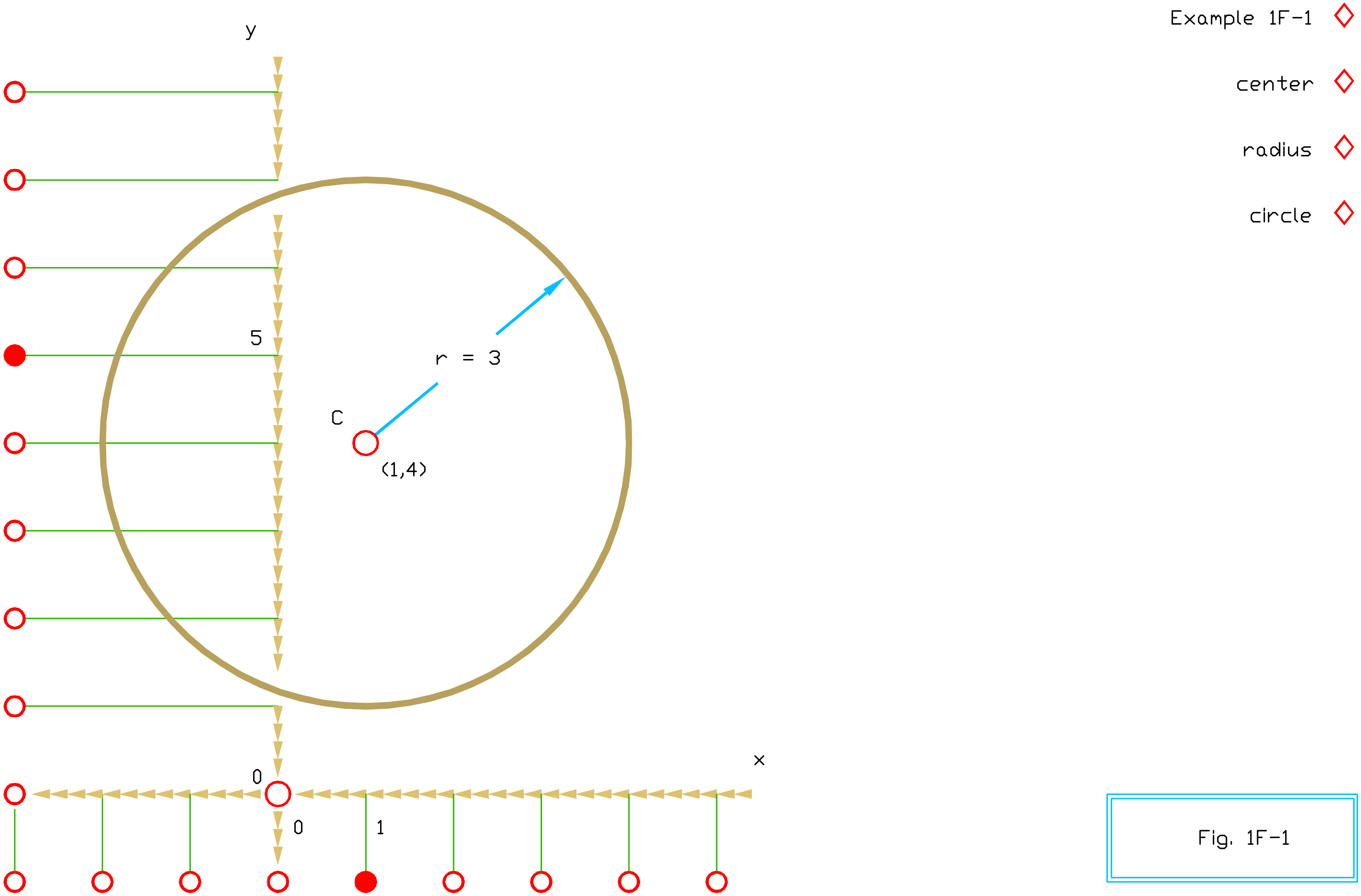


Fig. 1F-1

Figure 1F-1

The equation of a circle with center at $C = (a, b)$ and radius, r, is most conveniently presented in the center-radius form

$$(x - a)^2 + (y - b)^2 = r^2.$$

Every equation of the form

$$x^2 + y^2 + dx + gy + h = 0$$

is potentially the equation of a circle. In order to see this, the equation is rewritten in the center-radius form by a technique known as *completing the square*. For example, in order to sketch the graph of the equation,

$$x^2 + y^2 - 2x + 6y + 6 = 0,$$

we first compare it to the slightly expanded center-radius form,

$$(x^2 - 2ax + a^2) + (y^2 - 2by + b^2) = r^2.$$

Now, rewrite the given equation by collecting the terms in x and the terms in y, separately:

$$(x^2 - 2x) + (y^2 + 6y) = -6.$$

Notice, by comparing with the expanded center-radius form, that $a = 1$ and $b = -3$. Add the terms a^2 and b^2 to both sides of this equation to get

$$(x^2 - 2x + 1^2) + (y^2 + 6y + 3^2) = -6 + 1 + 9$$

or

$$(x - 1)^2 + (y + 3)^2 = 4 = 2^2.$$

This is the equation of the circle with center, $(a, b) = (1, -3)$, and radius, $r = 2$.

If, upon completing the square, the right side of the expanded center-radius form of the equation is negative, the equation cannot be satisfied since the left side is a sum of squares and cannot be negative. If the right side is zero, the circle is degenerate with radius zero and the equation is satified only by the point (a, b)

Example F-2. For the circle,

$$x^2 + y^2 - 4x + 2y - 4 = 0,$$

find the center C, radius r, and sketch.

Solution.
a. center-radius form: Complete the square as follows:

$$\begin{aligned}
x^2+y^2-4x+2y&=4\\
(x^2-4x)+(y^2+2y)&=4\\
(x^2-2\cdot 2x)+(y^2+2\cdot 1y)&=4\\
(x^2-2\cdot 2x+2^2)+(y^2+2\cdot 1y+1^2)&=4+4+1\\
(x-2)^2+(y+1)^2&=9.
\end{aligned}$$

b. Center and radius:

$$C=(2,1), \qquad r=3.$$

Example F-3. Find the equation of the circle that passes through the three points $A=(-3,3)$, $B=(1,5)$, $C=(4,4)$.

Solution.
a. System of equations for the unknown coefficients:
The equation of the circle has the general form

$$x^2+y^2+ux+vy+w=0,$$

where u, v, w are the unknown coefficients to be determined. Write this equation in the form

$$ux+vy+w=-(x^2+y^2).$$

Insert the coordinates of each of the points A, B, C here to get the system of three linear equations for u, v, w,

$$\begin{array}{lrl}
[\mathrm{A}] & -3u\ +3v+w= & -(3^2+3^2)=-18\\
[\mathrm{B}] & 1u\ +5v+w= & -(1^2+5^2)=-26\\
[\mathrm{C}] & 4u\ +4v+w= & -(4^2+4^2)=-32
\end{array}$$

b. Solve the system:

$$\begin{array}{rrrcr}
[\mathrm{D{=}B\text{-}A}] & 4u & +2v & = & -8\\
[\mathrm{E{=}C\text{-}B}] & 3u & -1v & = & -6\\
[\mathrm{F{=}2E}] & 6u & -2v & = & -12\\
[\mathrm{D{+}F}] & 10u & & = & -20
\end{array}$$

Hence, $u=-2$. From E, $v=0$. From A, $w=-24$.
c. Write the equation of the circle and put it in center-radius form:

$$x^2+y^2-2x-24=0$$

$$(x^2 - 2x) + (y^2 - 0y) = 24$$

$$(x^2 - 2 \cdot 1x + 1^2) + (y^2 - 2 \cdot 0y + 0^2) = 24 + 1 + 0$$

$$(x - 1)^2 + (y - 0)^2 = 25.$$

d. Sketch:
From the center-radius form, the center is $(1, 0)$ and the radius is 5.

Alternative method.
The center is the intersection of the perpendicular bisectors of AB and BC. The radius is the distance of any of the points A, B, C from the center.

EXERCISES 1.8
1. Find the equation of the circle with center C and radius r when C and r are, respectively,

a. $(1, 3)$, 2;	b. $(2, -3)$, 3;
c. $(0, 0)$, 1;	d. $(-1, 2)$, 2.

The circle in Part c is commonly called *the unit circle*.

2. Find the center and radius of each of the following circles. Sketch.

a. $x^2 + y^2 - 4x + y = -1/4$;	b. $x^2 + y^2 + 6x - 2y + 9 = 0$;
c. $x^2 + y^2 + 2x - 4y = 11$;	d. $x^2 + y^2 - 4x = 0$.

3. Find the equation of the circle passing through the points $(0, 1)$, $(0, 3)$, $(2, 5)$. Sketch.

ANSWERS

Exercises 1.3A

1. a. $2\sqrt{5}$, $1/2$. b. 5, $4/3$. c. $\sqrt{10}$, -3. d. 7, undefined.
e. 7, 0.

Exercises 1.3B

1. a. $y = x$. b. $y = 2$. c. $x = 3$. d. $y = 3 + x/2$.
e. $y = -2x - 2$.

Exercises 1.4

1. a. $y = x$. b. $x + y = 5$. c. $2x + y = 9$. d. $y = 2$.

Exercises 1.5

1. a. $x - 2y = -3$. b. $4x - 3y = -1$. c. $3x + y = 5$. d. $x = 1$.
e. $y = 3$.

Exercises 1.6

1. a. $(\frac{3}{2}, \frac{1}{2})$. b. parallel c. $(2, 3)$. d. $(1, 2)$.

Exercises 1.7

1. $3\sqrt{2}$.

2. $x + y = -3$.

3. $x + 2y = 5$.

4. $(3, 4)$.

Exercises 1.8

1. a. $x^2 + y^2 - 2x - 6y = -6$. b. $x^2 + y^2 - 4x + 6y + 4 = 0$.
c. $x^2 + y^2 = 1$. d. $x^2 + y^2 - 2x - 4y + 1 = 0$.

2. a. $(2, -1/2)$, 2. b. $(-3, 1)$, 1. c. $(-1, 2)$, 4. d. $(2, 0)$, 2.

3. $(x - 3)^2 + (y - 2)^2 = 10$.