ARE ALL ALGORITHMS FORMALLY UNDECIDABLE?

AN EXERCISE IN FORMAL LOGIC

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Is any algorithm ultimately provable?

This question was addressed by Kurt Godel in 1933 when his now famous proof showed the un-provability of any arithmetic algorithm and although the proof itself is very complicated, it has subsequently been refined by others and presented in more easily assimilated forms. Nevertheless the proof is essentially arithmetic in nature and the purpose of this paper is to examine as to whether or not the general principle of un-provability or more particularly un-decidabilty can be applied to any algebraic algorithm.

To begin this research which will be predicated on the theory of infinite classes and in particular on Russell's antimony which we can summarise as follows.

There are two fundamental classes, those which are members of themselves and those which are not members of themselves. Further we can define a class as being "normal" if and only if it does not contain itself as a member otherwise it will be called "non-normal".

Now, let "N" stand for the class of all normal classes and we ask:- Is "N" itself a normal class? If "N" is normal it is a member of itself (for by definition "N" contains all normal classes), but in that case "N" is non-normal because a class that contains itself as a member is non-normal. On the other hand, if "N" is non- normal it is a member of itself, but in that case

"N" is normal because the members of "N" are normal classes. In short "N" is normal if and only if "N" is non-normal. Thus the statement "N" is normal is both true and false and the statement is therefore formally un-decidable and therefore when we ask the question :- Is any algorithm ultimately provable we are really asking the question :- Is any algorithm ultimately un-decidable? And it is on the question of formal un-decidability that we shall continue this treatise.

The reasoning behind Russell's antimony or paradox as outlined above can be applied to number theory in general and can be written as a theorem in elementary logic as follows:-

$$(p \supset r) \supset [(q \supset r) \supset ((p \vee q) \supset r)]$$

which expresses a necessary truth and can be written in more conventional language as follows:-

If (p then r),then [if (if q then r) then (if (either p or q) then r)]

The foregoing statements can then be allocated to the components of Russell's antimony in the following manner.

Firstly, a normal class is a class which does not contain itself and in the case of number theory we can for example define any particular set of algorithms e.g:- $x^n + y^n = z^n$ as not being a member of itself because the set or class is not an algorithm and therefore the set of all possible algorithms is not a member of itself and we define this class as “p”.

Secondly, a non-normal class is a class which does contain itself and in this case we can define all possible numbers as being :-

$$-\infty \leftarrow 0 \rightarrow \infty \qquad (2)$$

Because the class of all possible numbers must contain itself and we define this class as “q”.

Thirdly we define “r” as the <u>sum</u> of all possible numbers as in (2) above which of course equates to zero.

Noting now that “p” contains all possible algorithms whether or not they have a solution and that this set must also sum to zero we can re-write (2) as:-

$$(p \supset 0) \supset [(q \supset 0) \supset ((p \vee q) \supset 0)]$$

Thus we conclude that since both “p” and “q” each sum to zero, that all algorithms are formally un-decidable.

Now again we ask the question “Is “N” itself a normal class? And we can repeat the answer given in Russell’s antimony i.e:- If “N” is normal it is a member of itself but in that case “N” is non-normal because a class that contains itself as a member is non-normal. But if “N” is non-normal it does contain itself, but in that case “N” is normal because the members of “N” are normal Thus “N” is normal if “N” is non-normal thus the statement “”N” is normal is both true and false and is therefore formally un-decidable and while we can show that in formal logic the underlying structure of our proposition is a necessary truth, the conclusions to be drawn from that truth are formally un-decidable.

The forgoing review does now raise further questions.

Firstly the normal class “p”, because it contains all possible algorithms it therefore contains all possible numbers and therefore could be defined as a non-normal class. Similarly the non-normal class “q” which is the sum of all possible numbers i.e:- $-\infty \leftarrow 0 \rightarrow \infty$ in fact sums to zero but the class also contains all possible algorithms and therefore the normal class is also non-normal, but a non-normal class equates to zero and therefore all possible algorithms can also be defined as zero and are therefore un-decidable.

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