Double and Half Angles

Derivation of Double angles for sine, cosine and tangent:

SINE

$$\sin 2\theta = \sin\theta\cos\theta + \sin\theta\cos\theta$$

you can simplify it as

$$\sin 2\theta = 2\sin\theta\sin\theta$$

(notice this only works for double angles*, this does not apply when sin has 2 different variables for example $\sin(\theta+\beta)$ or odd angles as $\sin 3\theta$)

COSINE

cos (A+A) = $\text{Cos}A^2$ - Sin^2A

cos (A+A) = (1-$\sin^2A$) - $\sin^2A$ ==> apply Pythagorean identities to eliminate a variable

= **1-2 $\mathbf{\sin^2 A}$**

(this is our final formula for cosine's double angle formula as sine being the variable)

OR

cos (A+A) = $\cos^2A$- (1-$\cos^2A$) ==> P.I.D

= **2 $\mathbf{\cos^2 A}$- 1**

(as cosine being the variable)

TANGENT

$$\tan 2(B) = \frac{(\tan B + \tan B)}{(1 - \tan B \tan B)} \text{ ==> Sum and of tangent formula}$$

$$= \frac{2\tan B}{(1 - \tan^2 B)} \text{ ==> Simplify}$$

Applying formulas:

$If\ \sin(b) = \frac{1}{2}$, use the double angle formula to find the exact value of

sin2(b),cos2(b),tan2(b) in the first quadrant

Step 1: find the value for cos(b) and tan(b) using pythagorean theorem

$$a^2 + b^2 = c^2$$

$$\sin(a) = \frac{\text{oppsite}}{\text{hypothenuse}} = \frac{1}{2} = \frac{a}{c}$$

$$2^2 = 1^2 + b^2$$

$$4 = 1 + b^2$$

$$3 = b^2$$

$$\pm\sqrt{3} = b$$

since it's in the first quadrant cos(a) must be postive

$$\cos(a) = \frac{c}{a} = \frac{\sqrt{3}}{2}$$

Step 2: find sin2(a)

$$\sin 2(a) = 2\sin a \cos a$$

$$\sin 2\left(\frac{1}{2}\right) = 2(\frac{1}{2})(\frac{\sqrt{3}}{2})$$

$$= \frac{\sqrt{3}}{2}$$

$Step 3$: find cos2(a)

$$\cos 2(a) = 1 - 2\sin^2 a$$

$$\cos 2(\frac{\sqrt{3}}{2}) = 1 - 2\left(\frac{1}{2}\right)^2$$

$$= 1 - 2(\frac{1}{4})$$

$$= 1 - (\frac{1}{2})$$

$$= -\frac{1}{2}$$

Step4: find tan2(a)

$$\tan(a) = \frac{\sin a}{\cos a} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{2} \times \frac{2}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\tan 2(a) = \frac{2\left(\frac{\sqrt{3}}{3}\right)}{1-\left(\frac{\sqrt{3}}{3}\right)^2}$$

$$= \frac{\frac{2\sqrt{3}}{3}}{\frac{9}{9}-\frac{3}{9}}$$

$$= \frac{\left(\frac{2\sqrt{3}}{3}\right)}{\left(\frac{6}{9}\right)}$$

$$= \left(\frac{2\sqrt{3}}{3}\right)\left(\frac{9}{6}\right)$$

$$= \frac{3\sqrt{3}}{3}$$

$$= \sqrt{3}$$

Half Angle

SINE

$$\cos 2\theta = 1 - 2\sin^2\theta$$

$$-1 \qquad -1$$

$$\cos 2\theta - 1 = -2\sin^2\theta$$

$$-(\cos 2\theta - 1) = 2\sin^2\theta -> \text{using distri. property on the left side}$$

$$\frac{(1-\cos 2\theta)}{2} = \frac{2\sin^2\theta}{2} => \text{divide both side by 2 (we are trying to isolate } \sin\theta)$$

$$\frac{(1-\cos 2\theta)}{2} = \sin^2\theta \;\; => \text{cross out 2 on the right side}$$

$$\sqrt{\frac{(1-\cos 2\theta)}{2}} = \sqrt{\sin^2\theta} \rightarrow \text{take the square root of both side}$$

$$\sqrt{\frac{(1-\cos 2\theta)}{2}} = \sin\theta \rightarrow \text{cancelation}$$

$$\text{NOTICE* since } \theta \text{ is arbitrary, we can replace } \theta \text{ for } \frac{\theta}{2} \; \textit{for} \text{ both side}$$

$$\sqrt{\frac{(1-\cos 2\frac{\theta}{2})}{2}} = \sin\frac{\theta}{2} \rightarrow \text{simplify by canceling 2 on the left side}$$

FINAL FORMULA

$$\sqrt{\frac{(1-\cos\theta)}{2}} = \sin\frac{\theta}{2}$$

COSINE

To do so, you can repeat the process that we did above except we use

$\cos 2\theta$ in term of cosine

$$\cos 2\theta = 2\cos^2\theta - 1$$

$+1 \qquad\qquad\qquad +1 \qquad \rightarrow$ Add 1 on both side

$$\frac{\cos 2\theta + 1}{2} = \frac{2\cos^2\theta}{2} \qquad \rightarrow \text{divide both side by 2}$$

$$\frac{\cos 2\theta + 1}{2} = \cos^2\theta \qquad \rightarrow \text{cross out the 2 on the right}$$

$$\sqrt{\frac{\cos 2\theta + 1}{2}} = \sqrt{\cos^2\theta} \qquad \rightarrow \text{take the square root}$$

$$\sqrt{\frac{\cos 2\theta + 1}{2}} = \cos\theta \qquad \rightarrow \text{cancelation}$$

Once again, we replace θ for $\frac{\theta}{2}$ (the angle is arbitrary)

since we are looking for the half angle formula.

$$\sqrt{\frac{\cos\theta + 1}{2}} = \cos\frac{\theta}{2}$$

TANGENT

$$\tan = \frac{\sin}{\cos} \qquad \rightarrow \text{tangent identity}$$

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$\tan\frac{\theta}{2} = \frac{\sqrt{\frac{(1-\cos\theta)}{2}}}{\sqrt{\frac{1+\cos\theta}{2}}} \qquad \rightarrow \text{half angle formula of sine and cosine}$$

$$step1 = \sqrt{\frac{(1-\cos\theta)}{2}} \times \sqrt{\frac{2}{1+\cos\theta}} \qquad \rightarrow \text{reciprocal to cancel the 2}$$

$$step2 = \sqrt{\frac{(1-\cos\theta)}{1+\cos\theta}} \times \sqrt{\frac{(1-\cos\theta)}{(1-\cos\theta)}} \qquad \rightarrow \text{multiply top and bottom to simplify}$$

$$step3 = \frac{1-\cos\theta}{\sqrt{1-\cos^2\theta}}$$

$$step4 = \frac{1-\cos\theta}{\sqrt{\sin^2\theta}} \qquad \rightarrow \text{Pythagorean identity } (\sin^2 A + \cos^2 A = 1)$$

$$step5 = \frac{1-\cos\theta}{\sin\theta} \qquad \rightarrow \text{cancel square root}$$

* we can take the square root of $1+\cos\theta$ instead of $\sqrt{1-\cos\theta}$

$$step2 = \sqrt{\frac{(1-\cos\theta)}{1+\cos\theta}} \times \sqrt{\frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \frac{\sqrt{1-\cos^2\theta}}{1+\cos\theta}$$

$$= \frac{\sqrt{\sin^2\theta}}{1+\cos\theta}$$

$$= \frac{\sin\theta}{1+\cos\theta}$$