Name KEY Date Period

Worksheet 4.4—Integration by *u*-Substitution and Pattern Recognition

Show all work. ***No calculator unless otherwise stated.***

Multiple Choice:

B 1. Find the most general function f such that $f''(x) = 9\cos 3x$

(A) $f(x) = -3\sin x + Cx^2 + D$ (B) $f(x) = -\cos 3x + Cx + D$ (C) $f(x) = -3\cos 3x + Cx^2 + D$ (D) $f(x) = \sin x + Cx + D$ (E) $f(x) = 3\sin 3x + Cx + D$

$f'(x) = (9)\left(\frac{1}{3}\right)\sin 3x + C$

$f'(x) = 3\sin 3x + C$

$f(x) = (3)\left(\frac{1}{3}\right)(-\cos 3x) + Cx + D$

$f(x) = -\cos 3x + Cx + D$

E 2. Evaluate the definite integral: $\int_0^1 \left(1 + e^{-x}\right)^2 dx$.

(A) $\frac{3}{2} - 2e + \frac{1}{2}e^2$ (B) $\frac{7}{2} + \frac{2}{e} + \frac{1}{2e^2}$ (C) $\frac{3}{2} - 2e - \frac{1}{2}e^2$ (D) $\frac{3}{2} + 2e + \frac{1}{2}e^2$ (E) $\frac{7}{2} - \frac{2}{e} - \frac{1}{2e^2}$

$\int_0^1 (1 + 2e^{-x} + e^{-2x})dx$ (expand)

$x - 2e^{-x} - \frac{1}{2}e^{-2x}\Big|_0^1$

$(1 - 2e^{-1} - \frac{1}{2}e^{-2}) - (0 - 2e^0 - \frac{1}{2}e^0)$

$1 - \frac{2}{e} - \frac{1}{2e^2} + 2 + \frac{1}{2}$

$\frac{7}{2} - \frac{2}{e} - \frac{1}{2e^2}$

A 3. Find the value of $f(-1)$ when $f'(x) = 6xe^{-2x^2}$, $f(0) = 1$.

(A) $\frac{5}{2} - \frac{3}{2}e^{-2}$ (B) $-\frac{3}{2}e^2$ (C) $-\frac{3}{2}e^{-2}$ (D) $\frac{5}{2} - \frac{3}{2}e^2$ (E) $\frac{5}{2} + \frac{3}{2}e^{-2}$

$f(-1) = f(0) + \int_0^{-1} f'(x)dx$

$= 1 + \int_0^{-1} 6xe^{-2x^2}dx$

$= 1 + (6)\left(-\frac{1}{4}\right)e^{-2x^2}\Big|_0^{-1}$

$= 1 - \frac{3}{2}\left[e^{-2} - e^0\right]$

$= 1 - \frac{3}{2e^2} + \frac{3}{2}$

$= \frac{5}{2} - \frac{3}{2e^2} = \frac{5}{2} - \frac{3}{2}e^{-2}$

$8-4x=2(4-2x)$ off by 2

A 4. Evaluate the definite integral $\int_0^1 (4-2x)e^{(8x-2x^2)}dx$

(A) $\frac{1}{2}(e^6-1)$ (B) $\frac{1}{2}(e^{-6}-1)$ (C) $e^{-6}+1$ (D) $\frac{1}{2}(e^6+1)$ (E) e^6-1

$\left(\frac{1}{2}\right)e^{(8x-2x^2)}\Big|_0^1$

$\frac{1}{2}\left[e^{(8-2)}-e^0\right]$

$\frac{1}{2}\left[e^6-1\right]$

A 5. Evaluate $\int_0^{\frac{\pi}{4}} \frac{2e^{\tan x}+5}{\cos^2 x}dx$

(A) $2e+3$ (B) $2e$ (C) $2e-3$ (D) e (E) $e+5$

$\int_0^{\pi/4}\left(2e^{\tan x}+5\right)(\sec^2 x)dx$

$\int_0^{\pi/4}\left(2e^{\tan x}\cdot\sec^2 x+5\sec^2 x\right)dx$

$2e^{\tan x}+5\tan x\Big|_0^{\pi/4}$

$\left(2e^{\tan \pi/4}+5\tan \pi/4\right)-\left(2e^{\tan 0}+5\tan 0\right)$

$2e+5-2e^0-0$

$2e+5-2$

$2e+3$

B 6. $\int \frac{4}{x}(1+2\ln x)^3\,dx =$

(A) $(1+2\ln x)^4+C$ (B) $\frac{1}{2}(1+2\ln x)^4+C$ (C) $-\frac{1}{2}(1+2\ln x)^4+C$ (D) $\frac{1}{2}\ln x(1+2\ln x)^4+C$ (E) $-(1+2\ln x)^4+C$

$2\left(\frac{1}{x}\right)$ off by 2

$4\int\left(\frac{1}{x}\right)\left(1+2\ln x\right)^3dx$

$(4)\left(\frac{1}{2}\right)\left(\frac{1}{4}\right)(1+2\ln x)^4+C$

rider corr rule

$\frac{1}{2}(1+2\ln x)^4+C$

B 7. Evaluate $\int_1^e \frac{1}{x}\left(f'(\ln x)+2\right)dx$ when $f(0)=1$ and $f(1)=4$.

(A) 6 (B) 5 (C) 4 (D) 3 (E) 2

$\int_1^e \left(\left(\frac{1}{x}\right)f'(\ln x) + 2\left(\frac{1}{x}\right)\right)dx$

$f(\ln x) + 2\ln|x| \Big|_1^e$

$\left(f(\ln e)+2(\ln e)\right) - \left(f(\ln 1)+2\ln 1\right)$

$f(1)+2 - f(0) - 0$

$4+2-1$

$6-1$

5

D 8. Evaluate $\int_0^1 \frac{6x}{1+x^2}dx$

(A) $\frac{3}{2}$ (B) 3 (C) 6 (D) $3\ln 2$ (E) $\frac{3}{2}\ln 2$

2x off by 2

$6\int_0^1 (x)(1+x^2)^{-1}dx$

$(6)\left(\frac{1}{2}\right)\ln|1+x^2|\Big|_0^1$

$3\left[\ln(2)-\ln(1)\right]$

$3[\ln 2 - 0]$

$3\ln 2$

B 9. Evaluate $\int_{\pi/4}^{3\pi/4} \frac{6\cos x - 2\sin x}{6\sin x + 2\cos x}dx$

(A) $-\ln\left(\frac{5}{2}\right)$ (B) $-\ln 2$ (C) $\ln\left(\frac{5}{2}\right)$ (D) $\ln 2$ (E) none of these

$6\cos x - 2\sin x$

$\int_{\pi/4}^{3\pi/4}(6\cos x - 2\sin x)(6\sin x + 2\cos x)^{-1}dx$

$\ln|6\sin x + 2\cos x|\Big|_{\pi/4}^{3\pi/4}$

$\ln\left|6\sin\frac{3\pi}{4}+2\cos\frac{3\pi}{4}\right| - \ln\left|6\sin\frac{\pi}{4}+2\cos\frac{\pi}{4}\right|$

$\ln\left|6\left(\frac{\sqrt{2}}{2}\right)+2\left(-\frac{\sqrt{2}}{2}\right)\right| - \ln\left|6\left(\frac{\sqrt{2}}{2}\right)+2\left(\frac{\sqrt{2}}{2}\right)\right|$

$\ln|3\sqrt{2}-\sqrt{2}| - \ln|3\sqrt{2}+\sqrt{2}|$

$\ln(2\sqrt{2}) - \ln(4\sqrt{2})$

$\ln\left(\frac{2\sqrt{2}}{4\sqrt{2}}\right)$ (condense)

$\ln\frac{1}{2}$

$-\ln 2$

B

10. Evaluate $\int_0^1 \frac{x^2+4x+1}{3x^2+3}dx$

(A) $\frac{1+4\ln 3}{6}$ (B) $\frac{1+2\ln 2}{3}$ (C) $\frac{2+4\ln 3}{3}$ (D) $\frac{1+2\ln 3}{3}$ (E) $\frac{1+4\ln 2}{6}$

$\frac{1}{3}\int_0^1 \frac{x^2+4x+1}{x^2+1}dx$ (factor out $\frac{1}{3}$)

Long division (deg 2 ÷ deg 2)

$$\begin{array}{r} 1 \\ x^2+1 \overline{\big)\, x^2+4x+1} \\ \underline{-x^2 \quad\quad +1} \\ 4x \end{array}$$

$\frac{1}{3}\int_0^1 \left(1+\frac{4x}{x^2+1}\right)dx$

$\frac{1}{3}\left[x+4\left(\frac{1}{2}\right)\ln|x^2+1|\right]\Big|_0^1$

$\frac{1}{3}\left[(1+2\ln 2)-(0+2\ln 1)\right]$

$\frac{1}{3}\left[1+2\ln 2\right]$

$\frac{1+2\ln 2}{3}$

E

11. Evaluate $\int_e^{e^4} \frac{5}{x\sqrt{\ln x}}dx$

(A) 6 (B) 7 (C) 8 (D) 9 (E) 10

$5\int_e^{e^4}\left(\frac{1}{x}\right)(\ln x)^{-1/2}$

$(5)(2)(\ln x)^{1/2}\Big|_e^{e^4}$

$10\left[\sqrt{\ln e^4}-\sqrt{\ln e}\right]$

$10\left[\sqrt{4}-\sqrt{1}\right]$

$10(2-1)$

10

Free Response:

12. Evaluate the following indefinite integrals. Don't forget your $+C$.

(a) $\int 2x\left(x^2+1\right)dx$

$\frac{1}{2}(x^2+1)^2+C$

(b) $\int \frac{3t^2}{t^3-4}dt$

$3\int t^2(t^3-4)^{-1}dt$

$3t^2$

$(3)\left(\frac{1}{3}\right)\ln|t^3-4|+C$

$\ln|t^3-4|+C$

(c) $\int x\sqrt{2x^2-1}dx$

$\int x(2x^2-1)^{1/2}dx$

$4x$

$\left(\frac{1}{4}\right)\left(\frac{2}{3}\right)(2x^2-1)^{3/2}+C$

$\frac{1}{6}\sqrt{(2x^2-1)^3}+C$

(d) $\int 3xe^{x^2+2}dx$

$3\int x\cdot e^{(x^2+2)}dx$

$2x$

$3\left(\frac{1}{2}\right)\cdot e^{(x^2+2)}+C$

$\frac{3}{2}e^{(x^2+2)}+C$

(e) $\int \frac{4x}{\left(x^2-8\right)^3}dx$

$4\int x\left(x^2-8\right)^{-3}dx$

$2x$

$(4)\left(\frac{1}{2}\right)\left(\frac{-1}{2}\right)\left(x^2-8\right)^{-2}+C$

rider corr rule

$\frac{-1}{\left(x^2-8\right)^2}+C$

(f) $\int 2re^{3r^2}dr$

$6r$

$2\int r\cdot e^{(3r^2)}dr$

$(2)\left(\frac{1}{6}\right)e^{3r^2}+C$

$\frac{1}{3}e^{3r^2}+C$

(g) $\int 5l^2\left(l^3-1\right)dl$

$5\int l^2\cdot\left(l^3-1\right)^1 dl$

$3l^2$

$(5)\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(l^3-1\right)^2+C$

$\frac{5}{6}\left(l^3-1\right)^2+C$

(h) $\int\left(3x^2+2\right)\sqrt{x^3+2x}\,dx$

$\int\left(3x^2+2\right)\left(x^3+2x\right)^{1/2}dx$

$3x^2+2$

$\frac{2}{3}\left(x^3+2x\right)^{3/2}+C$

(i) $\int\left(6t^2+10t^4\right)\left(t^3+t^5\right)^{100}dt$

$2\int\left(3t^2+5t^4\right)\left(t^3+t^5\right)^{100}dt$

$3t^2+5t^4$

$(2)\left(\frac{1}{101}\right)\left(t^3+t^5\right)^{101}+C$

$\frac{2}{101}\left(t^3+t^5\right)^{101}+C$

(j) $\int\frac{\ln^3 3x}{3x}dx$

$\frac{1}{3}\int\left(\frac{1}{x}\right)\left(\ln 3x\right)^3dx$

$\frac{1}{3x}\cdot 3=\frac{1}{x}$ (No corr!)

$\frac{1}{3}\left(\frac{1}{4}\right)\left(\ln 3x\right)^4+C$

$\frac{1}{12}\ln^4(3x)+C$

(k) $\int\frac{6x+5}{3x^2+5x-2}dx$

$\int(6x+5)\left(3x^2+5x-2\right)^{-1}dx$

$6x+5$

$\ln\left|3x^2+5x-2\right|+C$

(l) $\int\frac{12x+10}{9x^2+15x-6}dx$

$\frac{2}{3}\int(6x+5)\left(3x^2+5x-2\right)^{-1}dx$

$6x+5$

$\frac{2}{3}\ln\left|3x^2+5x-2\right|+C$

(m) $\int\frac{\cos 3x}{5+2\sin 3x}dx$

$\int(\cos 3x)\left(5+2\sin 3x\right)^{-1}dx$

$6\cos 3x$

$\frac{1}{6}\ln\left|5+2\sin 3x\right|+C$

(n) $\int(2t+1)e^{5t^2+5t}dt$

$10t+5=5(2t+1)$ off by 5

$\frac{1}{5}e^{(5t^2+5t)}+C$

(o) $\int\frac{\sin(\ln ax)}{x}dx$, where $a>0$

$\int\left(\frac{1}{x}\right)\sin\left(\ln ax\right)$

$\frac{1}{ax}\cdot a=\frac{1}{x}$ (No corr!)

$-\cos\left(\ln ax\right)+C$

(p) $\int\cos^3 t\,dt$

$\int\cos t\left(\cos^2 t\right)dt$

$\int\cos t\left(1-\sin^2 t\right)dt$

$\int\left(\cos t-(\cos t)(\sin t)^2\right)dt$

$\sin t-\frac{1}{3}(\sin t)^3+C$

$\sin t-\frac{1}{3}\sin^3 t+C$

13. Evaluate the following definite integrals without a calculator.

(a) $\int_0^1 x^3\left(1+x^4\right)^5 dx$

$4x^3$

$\left(\frac{1}{4}\right)\left(\frac{1}{6}\right)\left(1+x^4\right)^6\Big|_0^1$

$\frac{1}{24}\left[(1+1)^6-(1+0)^6\right]$

$\frac{1}{24}\left[2^6-1^6\right]$

$\frac{1}{24}\left[64-1\right]$

$\frac{63}{24}$

(b) $\int_{\sqrt{\pi/4}}^{\sqrt{2\pi/3}} x\sin\left(x^2\right)dx$

$2x$

$\left(\frac{1}{2}\right)\left(-\cos\left(x^2\right)\right)\Big|_{\sqrt{\pi/4}}^{\sqrt{2\pi/3}}$

$-\frac{1}{2}\left[\cos\left(\frac{2\pi}{3}\right)-\left(\cos\frac{\pi}{4}\right)\right]$

$-\frac{1}{2}\left[-\frac{1}{2}-\frac{\sqrt{2}}{2}\right]$

$\frac{1}{4}+\frac{\sqrt{2}}{4}$

$\frac{1+\sqrt{2}}{4}$

(c) $\int_{-1}^{3}\sqrt{7+3x}\,dx$

$\int_{-1}^{3}(7+3x)^{1/2}dx$

$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)(7+3x)^{3/2}\Big|_{-1}^{3}$

$\frac{2}{9}\left[(7+9)^{3/2}-(7-3)^{3/2}\right]$

$\frac{2}{9}\left[16^{3/2}-4^{3/2}\right]$

$\frac{2}{9}\left[4^3-2^3\right]$

$\frac{2}{9}\left[64-8\right]$

$\frac{2}{9}\left[56\right]$

$\frac{112}{9}$

(d) $\int_0^3 x\sqrt{1+x}\,dx$ u-sub required

$u=1+x$, $x=u-1$
$du=dx$

$\int_1^4(u-1)\left(u^{1/2}\right)du$

$\int_1^4\left(u^{3/2}-u^{1/2}\right)du$

$\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\Big|_1^4$

$\left(\frac{2}{5}\left(4^{5/2}\right)-\frac{2}{3}\left(4^{3/2}\right)\right)-\left(\frac{2}{5}-\frac{2}{3}\right)$

$\frac{2}{5}(32)-\frac{2}{3}(8)-\frac{2}{5}+\frac{2}{3}$

$\frac{64}{5}-\frac{16}{3}-\frac{2}{5}+\frac{2}{3}$

$\frac{62}{5}-\frac{14}{3}$

$\frac{186-70}{15}$

$\frac{116}{15}$

(e) $\int_0^\pi \cos^2\left(\frac{\theta}{5}\right)\sin\left(\frac{\theta}{5}\right)d\theta$

$\int_0^\pi\left(\cos\frac{\theta}{5}\right)^2\sin\left(\frac{\theta}{5}\right)d\theta$

$-\frac{1}{5}\sin\left(\frac{\theta}{5}\right)$

$(-5)\left(\frac{1}{3}\right)\left(\cos\frac{\theta}{5}\right)^3\Big|_0^\pi$

$-\frac{5}{3}\left[\left(\cos\frac{\pi}{5}\right)^3-(\cos 0)^3\right]$

$-\frac{5}{3}\left[\cos^3\frac{\pi}{5}-1\right]$

$\frac{5}{3}-\frac{5}{3}\cos^3\frac{\pi}{5}$

(f) $\int_0^1\frac{1+e^{3x}}{e^{3x}+3x}dx$

$\int_0^1\left(1+e^{3x}\right)\left(e^{3x}+3x\right)^{-1}dx$

$3+3e^{3x}=3\left(1+e^{3x}\right)$

$\left(\frac{1}{3}\right)\ln\left|e^{3x}+3x\right|\Big|_0^1$

$\frac{1}{3}\left[\ln\left(e^3+3\right)-\ln\left(e^0+0\right)\right]$

$\frac{1}{3}\left[\ln\left(e^3+3\right)-\ln 1\right]$

$\frac{1}{3}\ln\left(e^3+3\right)$

(g) $\int_0^1\frac{1}{1+9x^2}dx$ 3 (off by 3)

$a=1$ $u=3x$

$\left(\frac{1}{1}\right)\left(\frac{1}{3}\right)\arctan\left(\frac{3x}{1}\right)\Big|_0^1$

rule corr

$\frac{1}{3}\left[\arctan 3-\arctan 0\right]$

$\frac{1}{3}\left[\arctan 3-0\right]$

$\frac{1}{3}\arctan 3$

14. If $\int_a^b f(x)dx = K$, evaluate the following integrals in terms of K using your knowledge of transformations.

(a) $\int_{a+5}^{b+5} f(x-5)dx =$

Let $u = x - 5$

$du = dx, \ x = u + 5$

$\int_a^b f(u)\,du$

K

(b) $\int_a^b [f(x)+5]dx =$

$\int_a^b f(x)dx + \int_a^b 5\,dx$

$K + 5(b-a)$

(c) $\int_{a/5}^{b/5} f(5x)dx =$

Let $u = 5x$

$du = 5dx$

$dx = \frac{1}{5}du$

$\frac{1}{5}\int_a^b f(u)\,du$

$\frac{1}{5}K$

$\frac{K}{5}$

15. If $\int_3^6 f(z)dz = 4$, evaluate the following integrals exactly by using appropriate substitution and limits.

(a) $\int_{z=1}^{z=2} f(3z)dz$

Let $u = 3z$

$du = 3dz$

$dz = \frac{1}{3}du$

when $z=1, u=3$

when $z=2, u=6$

$\frac{1}{3}\int_3^6 f(u)\,du$

$\left(\frac{1}{3}\right)(4)$

$\frac{4}{3}$

(b) $\int_{0.5}^{2} f(7-2z)dz$

Let $u = 7 - 2z$

$du = -2dz$

$dz = -\frac{1}{2}du$

$-\frac{1}{2}\int_6^3 f(u)\,du$

$\frac{1}{2}\int_3^6 f(u)\,du$

$\frac{1}{2}(4)$

2

(c) $\int_4^7 (f(z-1)+5)dz$

Let $u = z - 1$

$\int_4^7 f(z-1)\,dz + \int_4^7 5\,dz$

$u = z-1$

$du = dz$

$\int_3^6 f(u)du + 5z\Big|_4^7$

$4 + 5[7-4]$

$4 + 5(3)$

$4 + 15$

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