3. $\frac{x^{3n}}{2n!+1} \le \frac{x^{3n}}{n!} \le \frac{(x^3)^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{(x^3)^n}{n!}$ is the Taylor series for e^{x^3} which converges for all x.

4. $\frac{x^{2n}}{n!+2} \le \frac{x^{2n}}{n!} \le \frac{(x^2)^n}{n!}$ and $\sum_{n=0}^{\infty} \frac{(x^2)^n}{n!}$ is the Taylor series for e^{x^2} which converges for all x.

5. $\left|\frac{(\cos x)^n}{n!+1}\right| \le \frac{|\cos x|^n}{n!} \le \frac{1}{n!}$ and $\sum_{n=0}^{\infty} \frac{1}{n!}$ converges to e.

6. $\left|\frac{2(\sin x)^n}{n!+3}\right| \le \frac{2|\sin x|^n}{n!} \le \frac{2}{n!}$ and $\sum_{n=0}^{\infty} \frac{2}{n!}$ converges to $2e$.

7. This is a geometric series which converges only for $|x| < 1$, so the radius of convergence is 1.

8. This is a geometric series which converges only for $|x+5| < 1$, so the radius of convergence is 1.

9. This is a geometric series which converges only for $|-(4x+1)| < 1$, or $\left|x+\frac{1}{4}\right| < \frac{1}{4}$, so the radius of convergence is $\frac{1}{4}$.

10. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|3x-2|^{n+1}}{n+1} \cdot \frac{n}{|3x-2|^n} = |3x-2|$

The series converges for $|3x-2| < 1$, or $\left|x-\frac{2}{3}\right| < \frac{1}{3}$, and diverges for $\left|x-\frac{2}{3}\right| > \frac{1}{3}$, so the radius of convergence is $\frac{1}{3}$.

11. This is a geometric series which converges only for $\left|\frac{x-2}{10}\right| < 1$, or $|x-2| < 10$, so the radius of convergence is 10.

12. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{n+3} \cdot \frac{n+2}{n|x|^n}$

$= |x| \cdot \lim_{n\to\infty} \frac{n^2+3n+2}{n^2+3n} = |x|$

The series converges for $|x| < 1$ and diverges for $|x| > 1$, so the radius of convergence is 1.

13. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)\sqrt{n+1}\ 3^{n+1}} \cdot \frac{n\sqrt{n}\,3^n}{|x|^n}$

$= \frac{|x|}{3} \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2} = \frac{|x|}{3}$

The series converges for $|x| < 3$ and diverges for $|x| > 3$, so the radius of convergence is 3.

14. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{2n+3}}{(n+1)!} \cdot \frac{n!}{|x|^{2n+1}} = \lim_{n\to\infty} \frac{x^2}{n+1} = 0$

The series converges for all values of x, so the radius of convergence is ∞.

15. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x+3|^{n+1}}{5^{n+1}} \cdot \frac{5^n}{n|x+3|^n}$

$= \frac{|x+3|}{5} \cdot \lim_{n\to\infty} \frac{n+1}{n} = \frac{|x+3|}{5}$

The series converges for $|x+3| < 5$ and diverges for $|x+3| > 5$, so the radius of convergence is 5.

16. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)|x|^{n+1}}{4^{n+1}[(n+1)^2+1]} \cdot \frac{4^n(n^2+1)}{n|x|^n}$

$= \frac{|x|}{4} \cdot \lim_{n\to\infty} \frac{n^3+n^2+n+1}{n^3+2n^2+2n} = \frac{|x|}{4}$

The series converges for $|x| < 4$ and diverges for $|x| > 4$, so the radius of convergence is 4.

17. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{(n+1)!|x-4|^{n+1}}{n!|x-4|^n}$

$= \lim_{n\to\infty} (n+1)|x-4|$

$= \infty \ (x \ne 4)$

The series converges only for $x = 4$, so the radius of convergence is 0.

18. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{\sqrt{n+1}\,|x|^{n+1}}{3^{n+1}} \cdot \frac{3^n}{\sqrt{n}\,|x|^n}$

$= \frac{|x|}{3} \cdot \lim_{n\to\infty} \sqrt{\frac{n+1}{n}}$

$= \frac{|x|}{3}$

The series converges for $|x| < 3$ and diverges for $|x| > 3$, so the radius of convergence is 3.

19. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|(-2)^{n+1}|(n+2)|x-1|^{n+1}}{|(-2)^n|(n+1)|x-1|^n}$

$= 2|x-1| \cdot \lim_{n\to\infty} \frac{n+2}{n+1}$

$= 2|x-1|$

The series converges for $|x-1| < \frac{1}{2}$ and diverges for $|x-1| > \frac{1}{2}$, so the radius of convergence is $\frac{1}{2}$.

20. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|4x-5|^{2n+3}}{(n+1)^{3/2}} \cdot \frac{n^{3/2}}{|4x-5|^{2n+1}}$

$= (4x-5)^2 \cdot \lim_{n\to\infty}\left(\frac{n}{n+1}\right)^{3/2}$

$= (4x-5)^2$

The series converges for $(4x-5)^2 < 1$, which is equivalent to $|4x-5| < 1$, or $\left|x - \frac{5}{4}\right| < \frac{1}{4}$ and diverges for $\left|x-\frac{5}{4}\right| > \frac{1}{4}$.

The radius of convergence is $\frac{1}{4}$.

21. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{|x+\pi|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x+\pi|^n}$

$= |x+\pi| \cdot \lim_{n\to\infty}\sqrt{\frac{n+1}{n}} = |x+\pi|$

The series converges for $|x+\pi| < 1$ and diverges for $|x+\pi| > 1$, so the radius of convergence is 1.

22. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{\left|x-\sqrt{2}\right|^{2n+3}}{2^{n+1}} \cdot \frac{2^n}{\left|x-\sqrt{2}\right|^{2n+1}}$

$= \lim_{n\to\infty}\frac{1}{2}(x-\sqrt{2})^2 = \frac{1}{2}(x-\sqrt{2})^2$

The series converges for $\frac{1}{2}(x-\sqrt{2})^2 < 1$, which is equivalent to $\left|x-\sqrt{2}\right| < \sqrt{2}$, and diverges for $\left|x-\sqrt{2}\right| > \sqrt{2}$. The radius of convergence is $\sqrt{2}$.

23. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x-1)^2}{4}$. It converges only when $\left|\frac{(x-1)^2}{4}\right| < 1$, so the interval of convergence is $-1 < x < 3$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x-1)^2}{4}} = \frac{4}{4-(x-1)^2} = \frac{4}{-x^2+2x+3} = -\frac{4}{x^2-2x-3}$$

24. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{(x+1)^2}{9}$. It converges only when $\left|\frac{(x+1)^2}{9}\right| < 1$, so the interval of convergence is $-4 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{(x+1)^2}{9}} = \frac{9}{9-(x+1)^2} = \frac{9}{-x^2-2x+8} = -\frac{9}{x^2+2x-8}$$

25. This is a geometric series with first term $a = 1$ and common ratio $r = \frac{\sqrt{x}}{2} - 1$. It converges only when $\left|\frac{\sqrt{x}}{2}-1\right| < 1$, so the interval of convergence is $0 < x < 16$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\left(\frac{\sqrt{x}}{2}-1\right)} = \frac{2}{4-\sqrt{x}}$$

26. This is a geometric series with first term $a = 1$ and common ratio $r = \ln x$. It converges only when $|\ln x| < 1$, so the interval of convergence is $\frac{1}{e} < x < e$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\ln x}$$

27. This is a geometric series with first term $a = 1$ and common ratio $\frac{x^2-1}{3}$. It converges only when $\left|\frac{x^2-1}{3}\right| < 1$, which is equivalent to $-2 < x^2 < 4$. It is always true that $x^2 > -2$, and $x^2 < 4$ implies that $|x| < 2$, so the interval of convergence is $-2 < x < 2$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{x^2-1}{3}} = \frac{3}{3-(x^2-1)} = \frac{3}{4-x^2}$$

28. This is a geometric series with first term $a = 1$ and common ratio $\frac{\sin x}{2}$. Since $\left|\frac{\sin x}{2}\right| < 1$ for all x, the interval of convergence is $-\infty < x < \infty$.

$$\text{Sum} = \frac{a}{1-r} = \frac{1}{1-\frac{\sin x}{2}} = \frac{2}{2-\sin x}$$

29. Diverges by the nth-Term Test, since $\lim_{n\to\infty}\frac{n}{n+1} = 1 \neq 0$.

30. Diverges by the nth-Term Test, since $\lim_{n\to\infty}\frac{2^n}{n+1} = \infty$. (The Ratio Test can also be used.)

31. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+1)^2-1}{2^{n+1}} \cdot \frac{2^n}{n^2-1} = \frac{1}{2} < 1.$$

32. Converges, because it is a geometric series with $r = \frac{1}{8}$, so $|r| < 1$.

33. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}}{(3^{n+1}+1)}\bullet\frac{3^n+1}{2^n}=\frac{2}{3}<1.$$

Alternate method: Note that $\frac{2^n}{3^n+1}<\left(\frac{2}{3}\right)^n$ for all n. Since

$\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, $\sum_{n=1}^{\infty}\frac{2^n}{3^n+1}$ converges by the Direct Comparison Test.

34. Diverges by the nth-Term Test, since

$$\lim_{n\to\infty}n\sin\left(\frac{1}{n}\right)=\lim_{n\to\infty}\frac{\sin(1/n)}{(1/n)}$$
$$=\lim_{x\to 0}\frac{\sin x}{x}\qquad\text{(where } x=1/n)$$
$$=1\neq 0$$

35. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^2e^{-n-1}}{n^2e^{-n}}=e^{-1}<1.$$

36. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)^{10}}{10^{n+1}}\bullet\frac{10^n}{n^{10}}=\frac{1}{10}<1.$$

37. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+4)!}{3!(n+1)!3^{n+1}}\bullet\frac{3!n!3^n}{(n+3)!}$$
$$=\lim_{n\to\infty}\frac{n+4}{3(n+1)}$$
$$=\frac{1}{3}<1.$$

38. Diverges by the nth-Term Test, since

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e\neq 0.$$

39. Converges, because it is a geometric series with $r=-\frac{2}{3}$, so $|r|<1$.

40. Diverges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!e^{-n-1}}{n!e^{-n}}=\lim_{n\to\infty}(n+1)e^{-1}=\infty.$$

(The nth-Term Test can also be used.)

41. Diverges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{3^{n+1}}{(n+1)^3 2^{n+1}}\bullet\frac{n^3 2^n}{3^n}$$
$$=\lim_{n\to\infty}\frac{3n^3}{(n+1)^3(2)}$$
$$=\frac{3}{2}>1.$$

(The nth-Term Test can also be used.)

42. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)\ln(n+1)}{2^{n+1}}\bullet\frac{2^n}{n\ln n}$$
$$=\lim_{n\to\infty}\frac{1}{2}\bullet\frac{n+1}{n}\bullet\frac{\ln(n+1)}{\ln n}$$
$$=\lim_{n\to\infty}\frac{1}{2}\bullet\frac{1}{1}\bullet\frac{1/(n+1)}{1/n}\qquad\text{(L'Hôpital's Rule)}$$
$$=\frac{1}{2}\bullet\lim_{n\to\infty}\bullet\frac{n}{n+1}=\frac{1}{2}<1$$

43. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{(2n+3)!}\bullet\frac{(2n+1)!}{n!}$$
$$=\lim_{n\to\infty}\frac{n+1}{(2n+3)(2n+2)}$$
$$=\lim_{n\to\infty}\frac{n+1}{4n^2+10n+2}$$
$$=\lim_{n\to\infty}\frac{1}{8n+10}\qquad\text{(L'Hôpital's Rule)}$$
$$=0<1$$

44. Converges by the Ratio Test, since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}}\bullet\frac{n^n}{n!}=\lim_{n\to\infty}\frac{(n+1)n^n}{(n+1)^{n+1}}$$
$$=\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n=\lim_{n\to\infty}\frac{1}{(1+1/n)^n}$$
$$=\frac{1}{e}<1$$

45. One possible answer: $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges (see Exploration 1 in this section) even though $\lim_{n\to\infty}\frac{1}{n}=0$.

46. One possible answer:

Let $a_n=2^{-n}$ and $b_n=3^{-n}$

Then $\sum a_n$ and $\sum b_n$ are convergent geometric series, but

$\sum\frac{a_n}{b_n}=\sum\left(\frac{3}{2}\right)^n$ is a divergent geometric series.

47. Almost, but the Ratio Test won't determine whether there is convergence or divergence at the endpoints of the interval.

48. $$\sum_{n=1}^{\infty}\frac{4}{(4n-3)(4n+1)}=\sum_{n=1}^{\infty}\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)$$

$$s_1=1-\frac{1}{5}$$
$$s_2=\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)=1-\frac{1}{9}$$
$$s_3=\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)=1-\frac{1}{13}$$
$$s_n=1-\frac{1}{4n+1}$$
$$S=\lim_{n\to\infty}s_n=1$$

49. $\sum_{n=1}^{\infty}\frac{6}{(2n-1)(2n+1)}=\sum_{n=1}^{\infty}\left(\frac{3}{2n-1}-\frac{3}{2n+1}\right)$

$s_1=3-\frac{3}{3}$

$s_2=(3-1)+\left(1-\frac{3}{5}\right)=3-\frac{3}{5}$

$s_3=(3-1)+\left(1-\frac{3}{5}\right)+\left(\frac{3}{5}-\frac{3}{7}\right)=3-\frac{3}{7}$

$s_n=3-\frac{3}{2n+1}$

$S=\lim_{n\to\infty}s_n=3$

50. $\frac{40n}{(2n-1)^2(2n+1)^2}=\frac{A}{(2n-1)^2}+\frac{B}{(2n+1)^2}$

$A(2n+1)^2+B(2n-1)^2=40n$

$n=-\frac{1}{2}\Rightarrow 4B=-20\Rightarrow B=-5$

$n=\frac{1}{2}\Rightarrow 4B=20\Rightarrow A=5$

$\sum_{n=1}^{\infty}\frac{40n}{(2n-1)^2(2n+1)^2}=\sum_{n=1}^{\infty}\left[\frac{5}{(2n-1)^2}-\frac{5}{(2n+1)^2}\right]$

$s_1=5-\frac{5}{9}$

$s_2=\left(5-\frac{5}{9}\right)+\left(\frac{5}{9}-\frac{5}{25}\right)=5-\frac{5}{25}$

$s_3=\left(5-\frac{5}{9}\right)+\left(\frac{5}{9}-\frac{5}{25}\right)+\left(\frac{5}{25}-\frac{5}{49}\right)=5-\frac{5}{49}$

$s_n=5-\frac{5}{(2n+1)^2}$

$S=\lim_{n\to\infty}s_n=5$

51. $\frac{2n+1}{n^2(n+1)^2}=\frac{A}{n^2}+\frac{B}{(n+1)^2}$

$A(n+1)^2+Bn^2=2n+1$

$n=0\Rightarrow A=1$

$n=-1\Rightarrow B=-1$

$\sum_{n=1}^{\infty}\frac{2n+1}{n^2(n+1)^2}=\sum_{n=1}^{\infty}\left(\frac{1}{n^2}-\frac{1}{(n+1)^2}\right)$

$s_1=1-\frac{1}{4}$

$s_2=\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{9}\right)=1-\frac{1}{9}$

$s_3=\left(1-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{16}\right)=1-\frac{1}{16}$

$s_n=1-\frac{1}{(n+1)^2}$

$S=\lim_{n\to\infty}s_n=1$

52. $s_1=1-\frac{1}{\sqrt{2}}$

$s_2=\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)=1-\frac{1}{\sqrt{3}}$

$s_3=\left(1-\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{3}}-\frac{1}{\sqrt{4}}\right)=1-\frac{1}{\sqrt{4}}$

$s_n=1-\frac{1}{\sqrt{n+1}}$

$S=\lim_{n\to\infty}s_n=1$

53. $s_1=\frac{1}{\ln 3}-\frac{1}{\ln 2}$

$s_2=\left(\frac{1}{\ln 3}-\frac{1}{\ln 2}\right)+\left(\frac{1}{\ln 4}-\frac{1}{\ln 3}\right)=\frac{1}{\ln 4}-\frac{1}{\ln 2}$

$s_3=\left(\frac{1}{\ln 3}-\frac{1}{\ln 2}\right)+\left(\frac{1}{\ln 4}-\frac{1}{\ln 3}\right)+\left(\frac{1}{\ln 5}-\frac{1}{\ln 4}\right)$

$=\frac{1}{\ln 5}-\frac{1}{\ln 2}$

$s_n=\frac{1}{\ln(n+2)}-\frac{1}{\ln 2}$

$S=\lim_{n\to\infty}s_n=-\frac{1}{\ln 2}$

54. $s_1=\tan^{-1}1-\tan^{-1}2=\frac{\pi}{4}-\tan^{-1}2$

$s_2=(\tan^{-1}1-\tan^{-1}2)+(\tan^{-1}2-\tan^{-1}3)$

$=\frac{\pi}{4}-\tan^{-1}3$

$s_3=(\tan^{-1}1-\tan^{-1}2)+(\tan^{-1}2-\tan^{-1}3)$

$+(\tan^{-1}3-\tan^{-1}4)$

$=\frac{\pi}{4}-\tan^{-1}4$

$s_n=\frac{\pi}{4}-\tan^{-1}(n+1)$

$S=\lim_{n\to\infty}s_n=\frac{\pi}{4}-\lim_{n\to\infty}\tan^{-1}n=\frac{\pi}{4}-\frac{\pi}{2}=-\frac{\pi}{4}$

55. True. See Theorem 8.

56. False. The power series

$\sum_{n=0}^{\infty}c_n(x-a)^n$ always converges at $x=a$. (The sum of the series is c_0.)

57. B. $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{2^{n+1}}{(-3)^{n+1}}\bullet\frac{(-3)^n}{2^n}=\lim_{n\to\infty}\frac{2}{-3}=-\frac{2}{3}$

58. C. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\frac{|2x-3|^{n+1}}{n+1}\bullet\frac{n}{|2x-3|^n}$

$=|2x-3|\bullet\lim_{n\to\infty}\frac{n}{n+1}$

$=|2x-3|$

The series converges for $|2x-3|<1$, which is equivalent to $\left|x-\frac{3}{2}\right|<\frac{1}{2}$, and diverges for $\left|x-\frac{3}{2}\right|>\frac{1}{2}$.

The radius of convergence is 1/2.

59. E. Consider first the series $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{|\sin x|^n}{2^n n^2}$

For any real number x, and for all n:

$$\frac{|\sin x|^n}{2^n n^2} \le \frac{1}{2^n n^2} < \frac{1}{2^n}$$

Since $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ converges, $\sum_{n=1}^{\infty}\frac{|\sin x|^n}{2^n n^2}$ converges by the Direct Comparison Test.

Then, by Theorem 8, since $\sum_{n=1}^{\infty}\frac{(\sin x)^n}{2^n n^2}$ converges absolutely, it also converges.

60. D. $\sum_{n=1}^{\infty}\frac{3}{(3n-1)(3n+2)} = \sum_{n=1}^{\infty}\frac{1}{3n-1} - \frac{1}{3n+2}$

$$s_1 = \frac{1}{2} - \frac{1}{5}$$

$$s_2 = \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) = \frac{1}{2} - \frac{1}{8}$$

$$s_3 = \left(\frac{1}{2}-\frac{1}{5}\right) + \left(\frac{1}{5}-\frac{1}{8}\right) + \left(\frac{1}{8}-\frac{1}{11}\right) = \frac{1}{2} - \frac{1}{11}$$

$$s_n = \frac{1}{2} - \frac{1}{3n+2}$$

$$S = \lim_{n\to\infty}\left(\frac{1}{2} - \frac{1}{3n+2}\right) = \frac{1}{2}$$

61. (a) For $k \le N$, it's obvious that

$$S_k = a_1 + \cdots + a_k \le a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n.$$

For all $k > N$,

$$\begin{aligned} S_k &= a_1 + \cdots + a_k = a_1 + \cdots + a_N + a_{N+1} + \cdots + a_k \\ &\le a_1 + \cdots + a_N + c_{N+1} + \cdots + c_k \\ &\le a_1 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n \end{aligned}$$

(b) Since all of the a_n are nonnegative, the partial sums of the series form a nondecreasing sequence of real numbers. Part (a) shows that this sequence is bounded above, so it must converge to a limit.

62. (a) For $k \le N$, it's obvious that

$$d_1 + \cdots + d_k \le d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n.$$

For all $k > N$,

$$\begin{aligned} &d_1 + \cdots + d_k = d_1 + \cdots + d_N + d_{N+1} + \cdots + d_k \\ &\le d_1 + \cdots + d_N + a_{N+1} + \cdots + a_k \\ &\le d_1 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n \end{aligned}$$

(b) If $\sum a_n$ converged, part (a) would imply that $\sum d_n$ was also convergent.

63. Answers will vary.

64. $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

Differentiate:

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} nx^{n-1}$$

Multiply by x:

$$\frac{x}{(1-x)^2} = \sum_{n=0}^{\infty} nx^n$$

Differentiate:

$$\begin{aligned} \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right) &= \frac{(1-x)^2(1) - (x)(2)(1-x)(-1)}{(1-x)^4} \\ &= \frac{(1-x)+2x}{(1-x)^3} = \frac{x+1}{(1-x)^3} \end{aligned}$$

$$\frac{x+1}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^{n-1}$$

Multiply by x:

$$\frac{x(x+1)}{(1-x)^3} = \sum_{n=0}^{\infty} n^2 x^n$$

Let $x = \frac{1}{2}$:

$$\frac{\frac{1}{2}\left(\frac{3}{2}\right)}{\left(\frac{1}{2}\right)^3} = \sum_{n=0}^{\infty} n^2\left(\frac{1}{2}\right)^n$$

$$6 = \sum_{n=0}^{\infty}\frac{n^2}{2^n}$$

The sum is 6.

Section 9.5 Testing Convergence at Endpoints (pp. 513–525)

Exploration 1 The p-Series Test

1. We first note that the Integral Test applies to any series of the form $\sum\frac{1}{n^p}$ where p is positive. This is because the function $f(x) = x^{-p}$ is continuous and positive for all $x > 0$, and $f'(x) = -p \cdot x^{-p-1}$ is negative for all $x > 0$.

If $p > 1$:

$$\begin{aligned} \int_1^{\infty}\frac{1}{x^p}\,dx &= \lim_{k\to\infty}\int_1^k \frac{1}{x^p}\,dx = \lim_{k\to\infty}\left(\left[\frac{x^{-p+1}}{-p+1}\right]_1^k\right) \\ &= \lim_{k\to\infty}\left(\frac{1}{1-p}\cdot\left(\frac{1}{k^{p-1}} - 1\right)\right) \\ &= 0 + \frac{1}{p-1} \quad (\text{since } p-1>0) \\ &= \frac{1}{p-1} < \infty. \end{aligned}$$

The series converges by the Integral Test.