PLEASE REFRAIN FROM USING INSTRUMENTS OF WEAKNESS!

From the 2003 BC Exam:

1. The function *f* has a Taylor series about x = 2 that converges to $f(x)$ for all x in the interval of convergence. The *nth* derivative of *f* at x = 2 is given by the following
$f^{(n)}(2) = \dfrac{(n+1)!}{3^n}$ for $n \geq 1$ and $f(2) = 1$.

(a) Write the first four terms & the general term of the Taylor Series for *f* about x = 2.

(b) Find the radius of convergence for the Taylor Series for *f* about x = 2. **Show the work that leads to your answer!**

(c) Let *g* be a function satisfying $g(2) = 3$ and $g'(x) = f(x)$ for all x. Write the first four terms and the general term of the Taylor Series for *g* about x = 2.

(d) Does the Taylor Series for *g* as defined in part (c) converge at x = -2? **Give a reason for your answer!**

From the 1999 BC Exam:

2. Let *f* be a function that has derivatives of all orders for all real numbers. Assume $f(0)=5,\ f'(0)=-3,\ f''(0)=1,\ f'''(0)=4$

a) Write the third degree Taylor Polynomial for *f* about $x=0$ and use it to approximate $f(0.2)$.

b) Write the fourth degree Taylor Polynomial for g, where $g(x)=f(x^2)$, about $x=0$.

c) Write the third degree Taylor Polynomial for *h*, where $h(x)=\int_0^x f(t)dt$, about $x=0$.

d) Let *h* defined as in part c). Given that $f(1)=3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

3. a. Find a power series for $f(x)=\dfrac{3}{x+2}$ centered at 0.

   b. Find the radius & interval of convergence for the above series.

4. a. Build a MacLaurin expansion for $f(x) = \sin\left(\sqrt{x}\right)$
   b. For what values of x does this series converge?
   c. Use your series from problem 3 to determine how many terms are needed to approximate $f(1.5)$ with error less than 10^{-6}

5. Determine whether the following converge or diverge **– state what test you're using and find an upper bound if possible!**

a. $$\sum_{n=o}^{\infty} \frac{2^n}{3^n + 2}$$

b. $$\sum_{n=1}^{\infty} \frac{n^{3/2} + 5}{n^2 + \sqrt{n}}$$

c. $$\sum_{n=1}^{\infty} \frac{\ln n^2}{n}$$

6. Consider the sequence $\left\{\left(1+\dfrac{1}{5n}\right)^{4n}\right\}$

a. Graph using $95 \le n \le 100$. Indicate the coordinate of $a_{95}..a_{100}$ rounded to four places

b. Notice your sequence seems to converge? Find the actual value the sequence converges to using analytic means and show work!

7. Recall the formal definition of a convergent sequence a.k.a. $\{a_n\} \to L$

For any $\varepsilon > 0,\ \exists\ M(\varepsilon) \ni$ if $n > M(\varepsilon)$ then $|a_n - L| < \varepsilon$

Find such a $M(\varepsilon)$ for the sequence $\left\{\dfrac{3n^2+1}{2n^2}\right\}$

8. Which of the following statements about series is true?

(A) If $\lim_{n\to\infty} u_n = 0$ then $\sum u_n$ converges.

(B) If $\lim_{n\to\infty} u_n \neq 0$ then $\sum u_n$ diverges.

(C) If $\sum u_n$ diverges then $\lim_{n\to\infty} u_n \neq 0$.

(D) (A) & (B)

(E) (B) & (C)

9. Which of the following converges **absolutely**?

(A) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$

(B) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$

(C) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

(D) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n+1}}$

(E) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+1)^2}$

10. Which of the following statements about series is false?

(A) If $\sum u_n$ converges, so does $\sum cu_n$ if $c \neq 0$

(B) If $\sum a_n$ and $\sum b_n$ converge, so does $\sum (ca_n + b_n)$, where $c \neq 0$

(C) If $\sum u_n$ converges, so does $\sum |u_n|$

(D) If $\sum |u_n|$ converges, so does $\sum u_n$

(E) Rearranging the terms of a positive convergent series will not affect its convergence or its sum

11. Which of the following converge?

(A) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{n}\right)$

(B) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

(C) $\sum_{n=1}^{\infty} \sin\left(\frac{1}{\sqrt{n}}\right)$

(D) $\sum_{n=1}^{\infty} \tan\left(\frac{1}{\sqrt{n}}\right)$

(E) None of the above

12. The series $\sum_{n=1}^{\infty} (-1)^{n-1}\left(\sqrt{n+1} - \sqrt{n}\right)$

(A) converges absolutely

(B) converges conditionally

(C) diverges to ∞

(D) diverges to -∞

(E) None of the above

13. The Alternating Series Remainder Theorem can be applied to each of the following EXCEPT:

(A) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt[3]{n+1}}$

(B) $\sum_{n=0}^{\infty} \frac{\cos(n\pi)}{n+1}$

(C) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\ln(n+1)}$

(D) $\sum_{n=0}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{\sqrt{n+1}}$

(E) $\sum_{n=0}^{\infty} \frac{\cos(2n\pi)}{n+1}$

3. Let f be a function that has derivatives of all orders for all real numbers. Assume $f(0) = 5$, $f'(0) = -3$, $f''(0) = 1$, and $f'''(0) = 4$.

(a) Write the third-degree Taylor polynomial for f about $x = 0$ and use it to approximate $f(0.2)$.

(b) Write the fourth-degree Taylor polynomial for g, where $g(x) = f(x^2)$, about $x = 0$.

(c) Write the third-degree Taylor polynomial for h, where $h(x) = \int_0^x f(t)\,dt$, about $x = 0$.

(d) Let h be defined as in part (c). Given that $f(1) = 3$, either find the exact value of $h(1)$ or explain why it cannot be determined.

(a) $P_3(f)(x) = 5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$

$f(0.2) \approx P_3(f)(0.2) =$

$5 - 3(0.2) + \frac{0.04}{2} + \frac{2(0.008)}{3} =$

4.425

3 { 2: $5 - 3x + \frac{1}{2}x^2 + \frac{2}{3}x^3$
<−1> each incorrect term, extra term, or $+\cdots$
1: approximates $f(0.2)$ }

<−1> for incorrect use of =

(b) $P_4(g)(x) = P_2(f)(x^2) = 5 - 3x^2 + \frac{1}{2}x^4$

2: $P_2(f)(x^2)$
<−1> each incorrect or extra term

(c) $P_3(h)(x) = \int_0^x \left(5 - 3t + \frac{1}{2}t^2\right) dt$

$= \left[5t - \frac{3}{2}t^2 + \frac{1}{6}t^3\right]_0^x$

$= 5x - \frac{3}{2}x^2 + \frac{1}{6}x^3$

2 { 1: $P_3(h)(x) = \int_0^x P_2(f)(t)\,dt$
1: answer
0/1 if any incorrect or extra terms }

(d) $h(1) = \int_0^1 f(t)\,dt$

cannot be determined because $f(t)$ is known only for $t = 0$ and $t = 1$

2 { 1: $h(1)$ cannot be determined
1: reason }