Vol 8. No. 1 Issue 2 – May, 2015

African Journal of Computing & ICT

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The Numerical Solution Of Stiff Problems In Ordinary Differential Equation (ODE) By Linear Multistep Methods

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ABSTRACT

Abstract: This paper is able to successfully use some linear Multistep methods and even a one-step method (Runge-Kutta IV) to the solution of Stiff problems in ordinary differential equation . The various solutions as represented in specific table confirms the theorem of Lambert 1973 on stiff problems as presented by initial value problems IVP

Keywords*:* Linear Multistep method, Adams bash forth, Adams Moulton method, Milne's method .true solution, ODE., Stiff problem, IVP.

African Journal of Computing & ICT Reference Format:
E.O. Ademola (2015): The Numerical Solution Of Stiff Problems In Ordinary Differential Equation (ODE) By Linear Multistep Methods. Afr J. of Comp & ICTs. Vol 8, No. 1, Issue 1. Pp 45-52

1. INTRODUCTION

Lambert 1973[4,5,7] defines stiffness as if Numerical method with a finite region of absolute stability applied to a system with any initial condition is forced to use in a certain interval of integration, a step length which is expressively small in relation to the smoothness of the exact solution in that interval, then the system is said to be stiff in the interval.

We also understand from [1] that a linear constant coefficient system is stiff if all of its eigenvalue have real negative real part and the stiffness ratio is large and finally stiffness occurs when some component of the solution decay much more rapidly than others

Significantly difficulties can occur when standard Numerical techniques are applied to approximate the solution of a differential equation when the exact solution contains terms of the form $e^{\lambda t}$ where λ is a complex number with negative real part

In the paper we shall be confirming the theorem of Lambert on stiff problem using some Numerical schemes the linear Multistep method (LMM) [2,9] which have been a very popular and powerful tool for solving initial value problem IVP of ordinary differential equation ode. Linear multistep method can as well be applied to higher order ODEs [2]. It is a known fact the LMM are non-self starting , hence need starting value from one-step method like Euler's , Runge-kutta and other one-step method family

The general n-step LMM is as given by Lambert (1973)[4]

$$\sum_{j=0}^{n} \alpha_j \, y_{k+j} \, h \sum_{j=0}^{n} \beta_j f_{n+j} \qquad (1)$$

Where α_j and β_j are uniquely determined and

$$\alpha_0 + \beta_0 \neq 0, \alpha_n = 1$$

The LMM is equation (1) generate discrete schemes which are used to solve first order ODEs, other researchers have introduced the continuous LMM using the continuous collocation and interpolation approach leading to the development of the continuous LMM of the form

$$y(x) = \sum_{j=0}^{n} \alpha_j(x) y_{k+j} \, h \sum_{j=0}^{n} \beta_j(x) f_{n+j} \qquad (2)$$

α_j and β_j are expressed as continuous function of x and at least differentiable one

But in this paper we shall be looking at several LMM(Adams Moulton, Milne's method) , Runge-kutta method of order IV a in the solution of ordinary differential stiff equations of the general form:

$$y' = Ay + f(x); y(x_0) = \eta; a \leq x \leq b, y \qquad (3)$$

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$f \in \Re^n$ *and A is a constant* $n \times n$ *matrix with eigenvalue* $\lambda_t \in \mathbb{C}, t = 1,2,\ldots,n$

The Adams-Bash forth-Moulton method and Milne's method uses $y_{n-3}, y_{n-2}, y_{n-1}$, and y_n in the calculation of y_{n+1}.This method is not self-starting; four points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, and (x_3, y_3) must be given in advance in order to generate the points $\{(x_n, y_n)\}_{n=4}^{m}$.[10,11,12].

A desirable feature of a multistep method is that the local truncation error (L.T.E) can be determined and the correction term can be included, which improves the accuracy of the answer at each step. Also, it is possible to determine if the step size is small enough to obtain an accurate value for y_{n+1}, yet large enough so that unnecessary and time-consuming calculations are eliminated. If the code for the subroutine is fine-tuned, then the combination of a predictor (Adams-Bash forth) and corrector (Adams-Moulton) requires only two function evaluations of $f(x, y)$ per step.

2. DERIVATION OF ADAMS-MOULTON METHOD

The Adams-Moulton method (also called the Adams-Bash forth-Moulton Corrector method when used in tandem with Adams-Bash forth method as a predictor-corrector pair) is a multistep method derived from the fundamental theorem of calculus given as:[6,7]

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f(x, y(x))dt \qquad (4)$$

Replace $y(x_{n+1})$, $y(x_n)$ and $f(x, y(x))$ with y_{n+1}, y_n and $f(x,y)$ respectively in equation (4) yield the numerical equivalent of the above formula given as;

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)dt \qquad (5)$$

Consider Newton's backward difference interpolation formula given as;

$$f(x,y) = f_n + u\nabla f_n + \frac{u(u+1)}{2}\nabla^2 f_n + \frac{u(u+1)(u+2)}{6}\nabla^3 f_n + \cdots$$
(6)

Where $u = \frac{x - x_0}{h}$ and $f_n = f(x_n, y_n)$

On substituting (6) into (5) we get;

$$y_{n+1} = y_n \int_{x_n}^{x_{n+1}} \left[f_n + u\nabla f_0 + \frac{u(u+1)}{2}\nabla^2 f_0 + \frac{u(u+1)(u+2)}{6}\nabla^3 f_n + \cdots\right] dx$$

$$= y_n + h\int_n^{n+1}\left[f_n + u\nabla f_n + \frac{u(u+1)}{2}\nabla^2 f_n + \frac{u(u+1)(u+2)}{6}\nabla^3 f_n + \cdots\right] du$$

$$= y_n + h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \cdots\right] f_n$$

Here one can see that right side of the above relation depends only on $y_n, y_{n-1}, y_{n-2}, \ldots$, all of which are known. Hence this formula can be used to compute y_{n+1} . Therefore, we can write it as;

$$y_{n+1}^p = y_n + \left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \cdots\right] f_n \qquad (7)$$

Equation (7) is called the Adams-Bash forth formula and is used as a predictor formula. (The superscript p indicating that it is a predicted value).

The corrector formula can be derived in a similar manner by using Newton's backward difference formula at f_{n+1}, that is,

$$f(x,y) = f_{n+1} + u\nabla f_{n+1} + \frac{u(u+1)}{2}\nabla^2 f_{n+1} + \frac{u(u+1)(u+2)}{3}\nabla^3 f_{n+1} + \cdots \qquad (8)$$

On substituting the value of $f(x,y)$ into (8) we get;

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}}\left[f_{n+1} + u\nabla f_{n+1} + \frac{u(u+1)}{2}\nabla^2 f_{n+1} + \frac{u(u+1)(u+2)}{3}\nabla^3 f_{n+1} + \cdots\right] dx$$

$$= y_n + h\int_{n+1}^{n}\left[f_{n+1} + u\nabla f_{n+1} + \frac{u(u+1)}{2}\nabla^2 f_{n+1} + \frac{u(u+1)(u+2)}{6}\nabla^3 f_{n+1} + \cdots\right] du$$

$$= y_n + h\left|1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \cdots\right| f_{n+1} \qquad (9)$$

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The right hand side of equation (9) depends upon $y_{n+1}, y_n, y_{n-1}, \ldots$ where for y_{n+1} we use y_{n+1}^p, the predicted value obtained from (7). The new value of y_{n+1} then obtained from (8) is called the corrected value and hence we rewrite the formula as;

$$y_{n+1}^c = y_n + h\left[1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \cdots\right] f_{n+1}^p \qquad (10)$$

This is called **Adams-Bash forth-Moulton** corrector formula or simply **Adams-Moulton** formula (The superscript 'c' indicating the corrected value of y).

For the sake of easy calculations, one can neglect the higher order differences and can express the lower order difference formula (9) and (10) can be written as;

$$y_{n+1}^p = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \qquad (11)$$

and

$$y_{n+1}^c = y_n + \frac{h}{24}[9f_{n+1}^p + 19f_n - 5f_{n-1} + f_{n-2}] \qquad (12)$$

In which the error are approximately $\frac{251}{720}h^5 f_n^{(4)}$ and $-\frac{19}{720}h^5 f_n^{(4)}$ respectively.

3. THEOREM 1.1 (ADAMS-BASH FORTH-MOULTON METHOD):[3]

Assuming that $f(x,y)$ is continuous and satisfies a Lipschits condition in the variable y, with consideration to the IVP(1) then, the Adams-Bash forth-Moulton method uses the formulas;

$x_{n+1} = x_n + h$,

$$y_{n+1}^p = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \quad \textbf{(The Predictor)} \qquad \textbf{(13)}$$

and

$$y_{n+1}^c = y_n + \frac{h}{24}[9f_{n+1}^p + 19f_n - 5f_{n-1} + f_{n-2}] \quad \textbf{(The Corrector)} \qquad (14)$$

for $n = 3, 4, \ldots m - 1$

as an approximate solution to the differential equation using the discrete set of points $\{(x_n, y_n)\}_{n=0}^m$.

4. THEOREM 1.2 (MILNE'S METHOD):[5,8]

Assuming that $f(x,y)$ is continuous and satisfies a Lipschits condition in the variable y, with consideration to the IVP(1) then, Milne's method uses the formulas;

$x_{n+1} = x_n + h$,

$$py_{n+1} = y_{n-3} + \frac{4h}{3}[2y'_n - y'_{n-1} + 2y'_{n-2}] \quad \textbf{(The Predictor)} \qquad \textbf{(15)}$$

and

$$y_{n+1} = y_{n-1} + \frac{h}{3}[py'_{n+1} + 4y'_n + y'_{n-1}] \quad \textbf{(The Corrector)} \qquad (16)$$

for $n = 3, 4, \ldots n - 1$

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5. NUMERICAL EXAMPLES

In this section, we will apply the linear multi step method, Adams moulton , Milne's and Runge-Kutta of order V to solve some IVP of ODEs. Errors associated with the methods are also obtained The fourth step Runge-kutta was used to obtained the starting values and the four stage Adams moulton and Milne's method was used as predator to the implicit schemes. The results and the error obtain are presented in tabular and graphical form.

EXAMPLE 1
Consider this Stiff equation [4,5]
$y' = -15y, y(0) = 1, a \leq x \leq b, y \in \Re$

Table 1.0

TABLE: RESULT OF Y'=-15Y, H=0.1				
X	**Exact Solution**	**Runge-Kutta**	**Milne's**	**Adam-Bash forth-Moulton**
0.0	1.0000000	1.0000000	1.0000000	1.0000000
0.1	0.2231302	0.2734375	0.2734375	0.2734375
0.2	0.0497871	0.0747681	0.0747681	0.0747681
0.3	0.0111090	0.0204444	0.0204444	0.0204444
0.4	0.0024788	0.0055903	0.0094910	-0.0853362
0.5	0.0005531	0.0015286	0.0025952	-0.1309541
0.6	0.0001234	0.0004180	0.0012407	-0.0029992
0.7	0.0000275	0.0001143	0.0074622	0.1150155
0.8	0.0000061	0.0000313	0.0010653	0.0062583
0.9	0.0000014	0.0000085	0.0075102	-0.1574968
1.0	0.0000003	0.0000023	0.0150121	-0.0540672

Table 1.2

COMPARISON OF ABSOLUTE ERROR H=0.1			
X	**Runge-kutta**	**Milne**	**Adam-Bash forth-Moulton**
0.0	0.0000000	0.0000000	0.0000000
0.1	0.0503073	0.0503073	0.0503073
0.2	0.0249810	0.0249810	0.0249810
0.3	0.0093354	0.0093354	0.0093354
0.4	0.0031115	0.0070122	0.0878149
0.5	0.0009755	0.0020421	0.1315071
0.6	0.0002946	0.0013642	0.0031226
0.7	0.0000868	0.0074347	0.1149879
0.8	0.0000251	0.0010591	0.0062522
0.9	0.0000072	0.0075115	0.1574982
1.0	0.0000020	0.0150118	0.0540675

Table 1.3

TABLE: RESULT OF Y'=-15Y, H=0.01				
X	**Exact Solution**	**Runge-Kutta**	**Milne**	**Adam-Bash forth-Moulton**
0.00	1.0000000	1.0000000	1.0000000	1.0000000
0.01	0.8607080	0.8607086	0.8607086	0.8607086
0.02	0.7408183	0.7408193	0.7408193	0.7408193
0.03	0.6376281	0.6376295	0.6376295	0.6376295
0.04	0.5488117	0.5488132	0.5488110	0.5488104
0.05	0.4723666	0.4723683	0.4723667	0.4723634
0.06	0.4065697	0.4065714	0.4065679	0.4065652
0.07	0.3499378	0.3499395	0.3499372	0.3499323
0.08	0.3011943	0.3011959	0.3011920	0.3011882
0.09	0.2592403	0.2592419	0.2592395	0.2592340
0.10	0.2231302	0.2231318	0.2231277	0.2231238

Table 1.4

COMPARISON OF ABSOLUTE ERROR H=0.01			
X	**Runge-Kutta**	**Milne**	**Adam-Bash forth-Moulton**
0.00	0.0000000	0.0000000	0.0000000
0.01	0.0000006	0.0000006	0.0000006
0.02	0.0000010	0.0000010	0.0000010
0.03	0.0000014	0.0000014	0.0000014
0.04	0.0000015	0.0000007	0.0000013
0.05	0.0000017	0.0000001	0.0000032
0.06	0.0000017	0.0000018	0.0000045
0.07	0.0000018	0.0000005	0.0000054
0.08	0.0000017	0.0000023	0.0000060
0.09	0.0000016	0.0000009	0.0000063
0.10	0.0000015	0.0000025	0.0000064

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Example 2
Consider IVP stiff problem [3]
$y' = -10y, y(0) = 2, a \leq x \leq b, y \in \Re$

Table2.1

TABLE: RESULT OF Y'=-10Y, H=0.1				
X	**Exact Solution**	**Runge-Kutta**	**Milne's**	**Adam-Bashforth-Moulton**
0.0	2. 00000000	2. 00000000	2.00000000	2.0000000
0.1	0.73575890	0.750000000	0.75000000	0.7500000
0.2	0.27067060	0.281250000	0.28125000	0.2812500
0.3	0.09957410	0.105468800	0.10546880	0.1054688
0.4	0.03663130	0.039550800	0.01562500	-0.0065308
0.5	0.01347590	0.014831500	0.01649310	-0.0343701
0.6	0.00495750	0.005561800	-0.00385800	-0.0021197
0.7	0.00182380	0.002085700	-0.01588750	0.0047212
0.8	0.00067090	0.000782100	0.01565590	-0.0127369
0.9	0.00024680	0.000293300	-0.01941580	-0.0084955
1.0	0.00009080	0.000110000	-0.00072790	0.0060342

Table 2.2

COMPARISON OF ABSOLUTE ERROR H=0.1			
X	**Runge-kutta**	**Milne**	**Adam-Bash forth-Moulton**
0.0	0.0000000	0.0000000	0.0000000
0.1	0.0142411	0.0142411	0.0142411
0.2	0.0105794	0.0105794	0.0105794
0.3	0.0058946	0.0058946	0.0058946
0.4	0.0029195	0.0210063	0.0431620
0.5	0.0013556	0.0030172	0.0478460
0.6	0.0006043	0.0088155	0.0070772
0.7	0.0002619	0.0177113	0.0028974
0.8	0.0001112	0.0149850	0.0134078
0.9	0.0000465	0.0196626	0.0087423
1.0	0.0000192	0.0008187	0.0059434

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Table 2.3

COMPARISON OF ABSOLUTE ERROR H=0.01			
X	**Runge-Kutta**	**Milne**	**Adam-Bash forth-Moulton**
0.00	0.0000000	0.0000000	0.0000000
0.01	0.0000001	0.0000001	0.0000001
0.02	0.0000002	0.0000002	0.0000002
0.03	0.0000004	0.0000004	0.0000004
0.04	0.0000006	0.0000001	0.0000002
0.05	0.0000006	0.0000001	0.0000008
0.06	0.0000006	0.0000005	0.0000013
0.07	0.0000007	0.0000002	0.0000017
0.08	0.0000007	0.0000007	0.0000020
0.09	0.0000007	0.0000004	0.0000021
0.10	0.0000007	0.0000008	0.0000023

TABLE: RESULT OF Y'=-10Y, H=0.001				
X	**Exact Solution**	**Runge-Kutta**	**Milne**	**Adam-Bash Moulton**
0.000	2.0000000	2.0000000	2.0000000	2.0000000
0.001	1.9800997	1.9800997	1.9800997	1.9800997
0.002	1.9603974	1.9603974	1.9603974	1.9603974
0.003	1.9408910	1.9408910	1.9408910	1.9408910
0.004	1.9215789	1.9215789	1.9215789	1.9215789
0.005	1.9024588	1.9024589	1.9024588	1.9024589
0.006	1.8835291	1.8835291	1.8835291	1.8835291
0.007	1.8647876	1.8647877	1.8647876	1.8647877
0.008	1.8462327	1.8462328	1.8462327	1.8462328
0.009	1.8278624	1.8278625	1.8278623	1.8278625

Table 2.5

COMPARISON OF ABSOLUTE ERROR H=0.001			
X	**Runge-Kutta**	**Milne**	**Adam-Bash forth-Moulton**
0.000	0.0000000	0.0000000	0.0000000
0.001	0.0000000	0.0000000	0.0000000
0.002	0.0000000	0.0000000	0.0000000
0.003	0.0000000	0.0000000	0.0000000
0.004	0.0000000	0.0000000	0.0000000
0.005	0.0000001	0.0000000	0.0000001
0.006	0.0000000	0.0000000	0.0000000
0.007	0.0000001	0.0000000	0.0000001
0.008	0.0000001	0.0000000	0.0000001
0.009	0.0000001	0.0000001	0.0000001

7.1 Discussion

The tables and the graphs above show the weakness of Linear multi step and Runge-kutta method at a large step length but at w a very small step length the rate of convergent of higher and better as they are compared with the true solution ..

8. CONCLUSION.

Linear multistep methods are not good numerical schemes to solve stiff problems as demonstrated on the tables ,but performs better with a very small step size which confirms the Lambert theorem. It was observed the Runge-kutta (one-step method) performed better than Linear multistep method.

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