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# The Solution Some Initial Value Problems In Ordinary Differential Equation (ODE) By Some Linear Multistep Method (LMM)

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**ABSTRACT**

In this paper we present the strength of some Linear multi-step method such as Adams –Bash forth, Adams- Moulton and Milne's method to the solution of Stiff problems in ODE. The solution is presented in Tabular formulae with it Numerical solution compared with the true (exact) solution and a corresponding graph is plotted using a mathematical software, with various mesh sizes.

**Keywords**: Linear Multistep method, Adams bash forth, Adams Moulton method, Milne's method .true solution, ode.

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## 1. INTRODUCTION

The linear Multistep method (LMM) have been a very popular and powerful tool for solving initial value problem IVP of ordinary differential equation ode. Linear multistep method can as well be applied to higher order ODEs[1]. It is a known fact the LMM are non-self starting , hence need starting value from one-step method like Euler's , Runge-kutta and other one-step method family[3].

The general n-step LMM is as given by Lambert (1973)[5]

$$\sum_{j=0}^{n} \alpha_j \, y_{k+j} \, h \sum_{j=0}^{n} \beta_j f_{n+j} \qquad (1)$$

Where
$\alpha_j$ and $\beta_j$ are uniquely determined and
$\alpha_0 + \beta_0 \neq 0, \alpha_n = 1$

The LMM is equation (1) generate discrete schemes which are used to solve first order ODEs, other researchers have introduced the continuous LMM using the continuous collocation and interpolation approach leading to the development of the continuous LMM of the form

$$y(x) = \sum_{j=0}^{n} \alpha_j \, (x) y_{k+j} \, h \sum_{j=0}^{n} \beta_j(x) f_{n+j} \qquad (2)$$

$\alpha_j$ and $\beta_j$ are expressed as continuous function of x and at least differentiable one

But in this paper we shall be looking at several LMM(Adams Moulton, Milne's method) , Runge-kutta method of order IV and IV ode stiff in the solution of ordinary differential equation of the general form:

$$y' = f(x, y); y(x_0) = \eta; a \leq x \leq b, y \in \Re \qquad (3)$$

The Adams-Bash forth-Moulton method and Milne's method uses $y_{n-3}, y_{n-2}, y_{n-1}$, and $y_n$ in the calculation of $y_{n+1}$.This method is not self-starting; four points $(x_0, y_0), (x_1, y_1), (x_2, y_2)$, and $(x_2, y_2)$ must be given in advance in order to generate the points $\{(x_n, y_n)\}_{n=4}^{m}$ .[3,4]

A desirable feature of a multistep method is that the local truncation error (L.T.E) can be determined and the correction term can be included, which improves the accuracy of the answer at each step. Also, it is possible to determine if the step size is small enough to obtain an accurate value for $y_{n+1}$, yet large enough so that unnecessary and time-consuming calculations are eliminated. If the code for the subroutine is fine-tuned, then the combination of a predictor (Adams-Bashforth) and corrector(Adams-Moulton) requires only two function evaluations of $f(x, y)$ per step.[5,7]

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## 2. DERIVATION OF ADAMS-MOULTON METHOD

The Adams-Moulton method (also called the Adams-Bash forth-Moulton Corrector method when used in tandem with Adams-Bash forth method as a predictor-corrector pair) is a multistep method derived from the fundamental theorem of calculus given as:[6,7]

$$y(x_{n+1}) = y(x_n) + \int_{x_n}^{x_{n+1}} f\big(x, y(x)\big)dt \qquad (4)$$

Replace $y(x_{n+1})$, $y(x_n)$ and $f\big(x, y(x)\big)$ with $y_{n+1}$, $y_n$ and $f(x,y)$ respectively in equation (4) yield the numerical equivalent of the above formula given as;

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(x,y)\,dt \qquad (5)$$

Consider Newton's backward difference interpolation formula given as;

$$f(x,y) = f_n + u\nabla f_n + \frac{u(u+1)}{2}\nabla^2 f_n + \frac{u(u+1)(u+2)}{6}\nabla^3 f_n + \cdots \qquad (6)$$

Where $u = \frac{x-x_0}{h}$ and $f_n = f(x_n, y_n)$

On substituting (6) into (5) we get;

$$y_{n+1} = y_n \int_{x_n}^{x_{n+1}} \left[f_n + u\nabla f_0 + \frac{u(u+1)}{2}\nabla^2 f_0 + \frac{u(u+1)(u+2)}{6}\nabla^3 f_n + \cdots\right]dx$$

$$= y_n + h\int_n^{n+1}\left[f_n + u\nabla f_n + \frac{u(u+1)}{2}\nabla^2 f_n + \frac{u(u+1)(u+2)}{6}\nabla^3 f_n + \cdots\right]du$$

$$= y_n + h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \cdots\right]f_n$$

Here one can see that right side of the above relation depends only on $y_n, y_{n-1}, y_{n-2}, \ldots,$ all of which are known. Hence this formula can be used to compute $y_{n+1}$. Therefore, we can write it as;

$$y_{n+1}^p = y_n + \left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \cdots\right]f_n \qquad (7)$$

Equation (7) is called the Adams-Bash forth formula and is used as a predictor formula. (The superscript $p$ indicating that it is a predicted value).

The corrector formula can be derived in a similar manner by using Newton's backward difference formula at $f_{n+1}$, that is,

$$f(x,y) = f_{n+1} + u\nabla f_{n+1} + \frac{u(u+1)}{2}\nabla^2 f_{n+1} + \frac{u(u+1)(u+2)}{3}\nabla^3 f_{n+1} + \cdots \qquad (8)$$

On substituting the value of $f(x,y)$ into (8) we get;

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}}\left[f_{n+1} + u\nabla f_{n+1} + \frac{u(u+1)}{2}\nabla^2 f_{n+1} + \frac{u(u+1)(u+2)}{3}\nabla^3 f_{n+1} + \cdots\right]dx$$

$$= y_n + h\int_{n+1}^{n}\left[f_{n+1} + u\nabla f_{n+1} + \frac{u(u+1)}{2}\nabla^2 f_{n+1} + \frac{u(u+1)(u+2)}{6}\nabla^3 f_{n+1} + \cdots\right]du$$

$$= y_n + h\left[1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \cdots\right]f_{n+1} \qquad (9)$$

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The right hand side of equation (9) depends upon $y_{n+1}, y_n, y_{n-1}, \ldots$where for $y_{n+1}$ we use $y^p_{n+1}$, the predicted value obtained from (7). The new value of $y_{n+1}$ then obtained from (8) is called the corrected value and hence we rewrite the formula as;

$$y^c_{n+1} = y_n + h\left[1 - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 - \frac{19}{720}\nabla^4 - \cdots\right] f^p_{n+1} \qquad (10)$$

This is called **Adams-Bash forth-Moulton** corrector formula or simply **Adams-Moulton** formula (The superscript '$c$' indicating the corrected value of $y$).

For the sake of easy calculations, one can neglect the higher order differences and can express the lower order difference formula (9) and (10) can be written as;

$$y^p_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \qquad (11)$$

and

$$y^c_{n+1} = y_n + \frac{h}{24}[9f^p_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \qquad (12)$$

In which the error are approximately $\frac{251}{720}h^3 f^{(4)}_n$ and $-\frac{19}{720}h^3 f^{(4)}_n$ respectively.

**3. THEOREM 1.1 (ADAMS-BASH FORTH-MOULTON METHOD):[3]**

Assuming that $f(x, y)$ is continuous and satisfies a Lipschits condition in the variable $y$, with consideration to the IVP(1) then, the Adams-Bashforth-Moulton method uses the formulas;

$$x_{n+1} = x_n + h,$$

$$y^p_{n+1} = y_n + \frac{h}{24}[55f_n - 59f_{n-1} + 37f_{n-2} - 9f_{n-3}] \quad \textbf{(The Predictor)} \qquad \textbf{(13)}$$

and

$$y^c_{n+1} = y_n + \frac{h}{24}[9f^p_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}] \quad \textbf{(The Corrector)}$$
(14)

for $n = 3,4, \ldots m - 1$

as an approximate solution to the differential equation using the discrete set of points $\{(x_n, y_n)\}_{n=0}^{m}$.

**4. THEOREM 1.2 (MILNE'S METHOD):[5,8]**

Assuming that $f(x, y)$ is continuous and satisfies a Lipschits condition in the variable $y$, with consideration to the IVP(1) then, Milne's method uses the formulas;

$$x_{n+1} = x_n + h,$$

$$py_{n+1} = y_{n-3} + \frac{4h}{3}[2y'_n - y'_{n-1} + 2y'_{n-2}] \qquad \textbf{(The Predictor)} \qquad \textbf{(13)}$$

and

$$y_{n+1} = y_{n-1} + \frac{h}{3}[py'_{n+1} + 4y'_n + y'_{n-1}] \qquad \textbf{(The Corrector)} \qquad (14)$$

for $n = 3,4, \ldots n - 1$

## 5. NUMERICAL EXAMPLES

In this section, we will apply the linear multi step method, Adams moulton , Milne's and Runge-Kutta of order V to solve some IVP of ODEs. Errors associated with the methods are also obtained The fourth step Runge-kutta was used to obtained the starting values and the four stage Adams moulton and Milne's method was used as predator to the implicit schemes. The results and the error obtain are presented in tabular and graphical form.

## 6. **EXAMPLE 1**.

Consider the IVP $y' = y \;\; 0 \leq x \leq 1, y(0) = 1$

The Exact solution $y(x) = e^{x}$

**TABLE 1: RESULT OF Y'=Y, H=0.1**

| Result of Y'=Y, H=0.1 | | | | |
|---|---|---|---|---|
| **X** | **Exact Solution** | **Runge-Kutta** | **Milne's Method** | **Adam-Bash forth-Moulton** |
| 0.0 | 1.0000000 | 1.0000000 | 1.0000000 | 1.0000000 |
| 0.1 | 1.1051710 | 1.1051708 | 1.1051708 | 1.1051708 |
| 0.2 | 1.2214028 | 1.2214025 | 1.2214025 | 1.2214025 |
| 0.3 | 1.3498589 | 1.3498584 | 1.3498584 | 1.3498584 |
| 0.4 | 1.4918247 | 1.4918242 | 1.4918244 | 1.4918244 |
| 0.5 | 1.6487212 | 1.6487205 | 1.6487209 | 1.6487211 |
| 0.6 | 1.8221189 | 1.8221178 | 1.8221184 | 1.8221189 |
| 0.7 | 2.0137527 | 2.0137515 | 2.0137522 | 2.0137532 |
| 0.8 | 2.2255411 | 2.2255394 | 2.2255404 | 2.2255418 |
| 0.9 | 2.4596033 | 2.4596014 | 2.4596026 | 2.4596045 |
| 1.0 | 2.7182822 | 2.7182798 | 2.7182813 | 2.7182837 |

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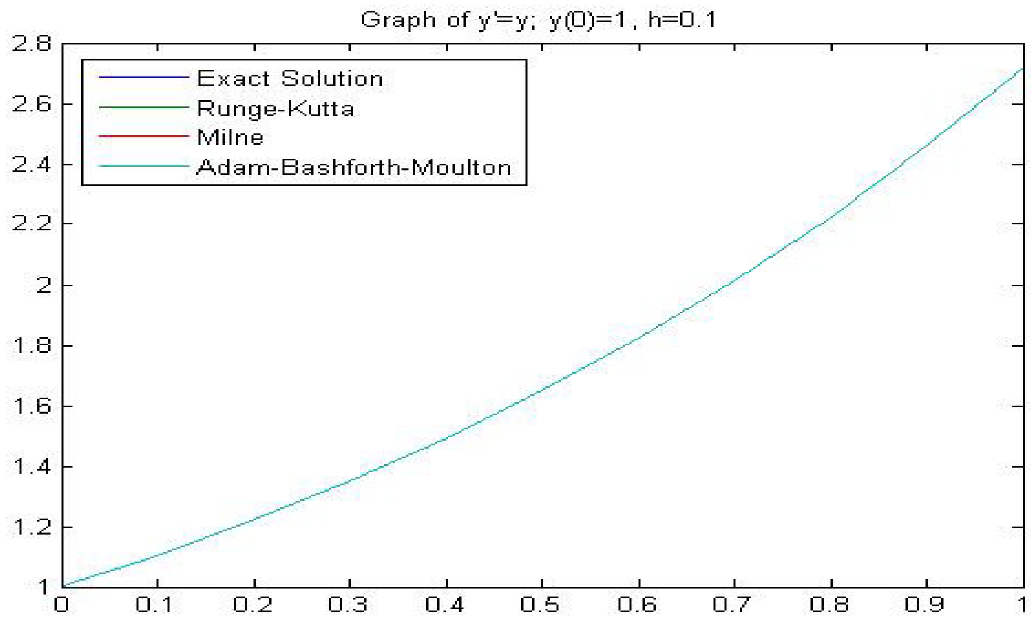


**Fig. 1: Graph of Result of Y'=Y, H=0.1**

**EXAMPLE 2**.

Consider the IVP $y' = y^2 + 1 \quad 0 \le x \le 1, y(0) = 0$

The Exact solution $y(x) = tanx$

**TABLE 2: RESULT OF Y'=Y^2+1, H=0.1**

| Result of Y'=Y^2+1, H=0.1 | | | | |
|---|---|---|---|---|
| **X** | **Exact Solution** | **Runge-Kutta** | **Milne** | **Adam-Bashforth-Moulton** |
| 0.0 | 0.0000000 | 0.0000000 | 0.0000000 | 0.0000000 |
| 0.1 | 0.1003347 | 0.1003346 | 0.1003346 | 0.1003346 |
| 0.2 | 0.2027100 | 0.2027099 | 0.2027099 | 0.2027099 |
| 0.3 | 0.30 93363 | 0.3093360 | 0.3093360 | 0.3093360 |
| 0.4 | 0.4227932 | 0.4227930 | 0.4227946 | 0.4227981 |
| 0.5 | 0.5463025 | 0.5463023 | 0.5463043 | 0.5463150 |
| 0.6 | 0.6841369 | 0.6841367 | 0.6841404 | 0.6841611 |
| 0.7 | 0.8422884 | 0.8422886 | 0.8422925 | 0.8423319 |
| 0.8 | 1.0296386 | 1.0296390 | 1.0296422 | 1.0297143 |
| 0.9 | 1.2601584 | 1.2601588 | 1.2601517 | 1.2602882 |
| 1.0 | 1.5574081 | 1.5574064 | 1.5573580 | 1.5576259 |

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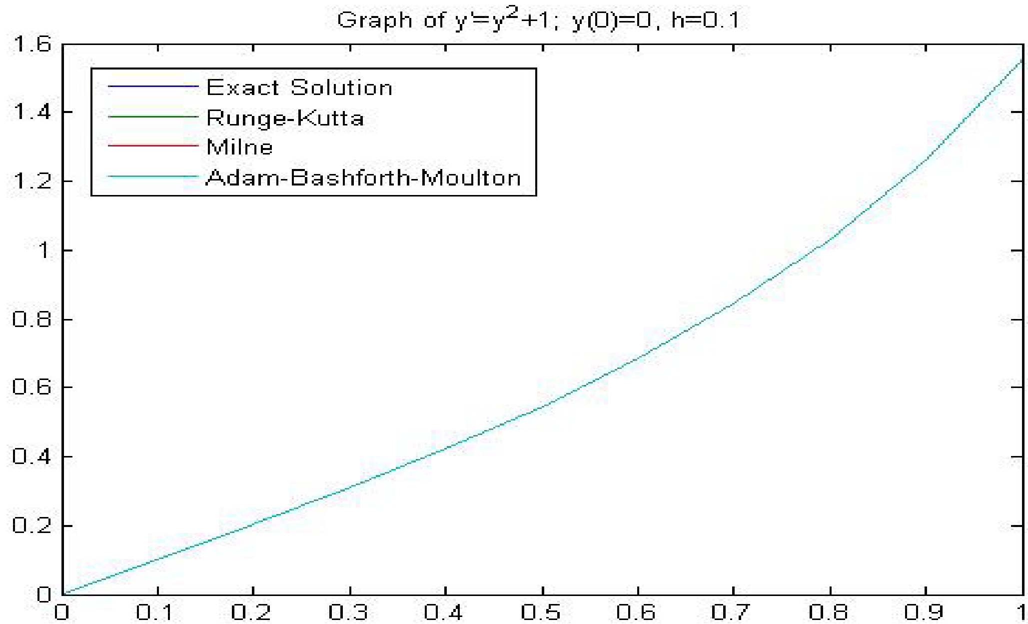


**Fig. 2: Graph of Result Of Y'=Y, H=0.1**

**EXAMPLE 3**.

Consider the IVP $y' = y - x \quad 0 \leq x \leq 1, y(0) = 2$

The Exact solution $y(x) = e^X + X + 1$

**TABLE 3: RESULT OF Y'=Y-X, H=0.1**

| Result of Y'=Y-X, H=0.1 | | | | |
|---|---|---|---|---|
| **X** | **Exact Solution** | **Runge-Kutta** | **Milne's** | **Adam-Bash forth-Moulton** |
| 0.0 | 2.0000000 | 2.0000000 | 2.0000000 | 2.0000000 |
| 0.1 | 2.2051709 | 2.2051709 | 2.2051709 | 2.2051709 |
| 0.2 | 2.4214027 | 2.4214027 | 2.4214027 | 2.4214027 |
| 0.3 | 2.6498590 | 2.6498587 | 2.6498587 | 2.6498587 |
| 0.4 | 2.8918247 | 2.8918245 | 2.8918247 | 2.8918247 |
| 0.5 | 3.1487212 | 3.1487210 | 3.1487212 | 3.1487215 |
| 0.6 | 3.4221189 | 3.4221184 | 3.4221189 | 3.4221191 |
| 0.7 | 3.7137527 | 3.7137520 | 3.7137527 | 3.7137535 |
| 0.8 | 4.0255413 | 4.0255399 | 4.0255413 | 4.0255423 |
| 0.9 | 4.3596034 | 4.3596020 | 4.3596034 | 4.3596048 |
| 1.0 | 4.7182822 | 4.7182803 | 4.7182822 | 4.7182841 |

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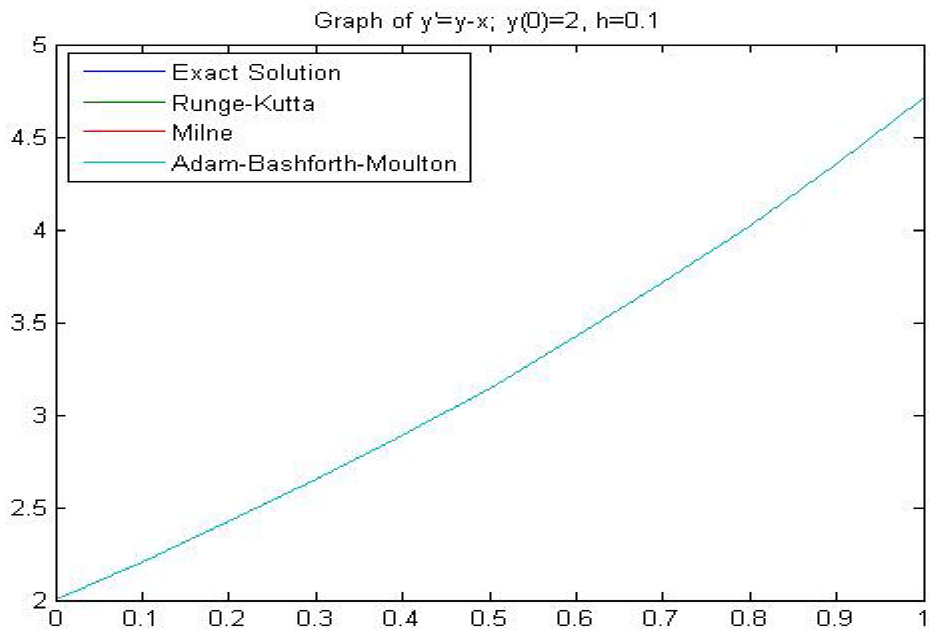


**Fig 3: RESULT OF Y'=Y-X, H=0.1**

## 7. DISCUSSION

The tables and the graphs above show the strength of Linear multi step and Runge-kutta method as they are compared with the true solution where the rate of convergent with the true solution is high.

## 8. CONCLUSION.

Linear multistep methods are good numerical tool that is convenient to solve initial value problems as demonstrated in this paper.

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