Section 2: Numerical integration

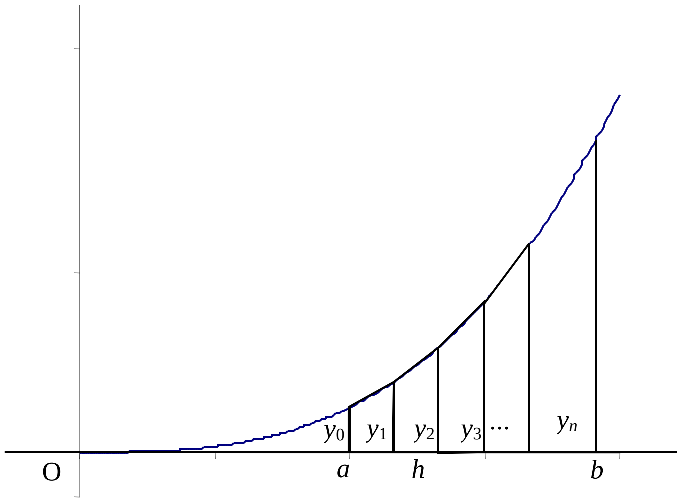
Notes and Examples

These notes contain subsections on:

- The trapezium rule
- Upper and lower bounds for integrals using rectangles

The trapezium rule

Not all functions can be integrated. For functions which cannot be integrated, you can find an approximate value for a definite integral by using a numerical method such as the trapezium rule. In this method, the area under the graph can be approximated by dividing it into a number of trapezia.



If the trapezia have width h, and the y-coordinates are $y_1, y_2, \ldots, y_n$, then the total area A is:

$$A \approx \frac{h}{2}\left[y_0 + y_n + 2\left(y_1 + y_2 + \ldots + y_{n-1}\right)\right] \qquad \text{where } h = \frac{(b-a)}{n}.$$

Example 1

(i) Use the trapezium rule with: (a) 3 strips and (b) 6 strips, to find the approximate value of $\int_0^3 \sqrt{1+x^3}\,dx$.

(ii) Explain why the trapezium rule is an underestimate of the true value.

Solution

(i) (a) With 3 strips of width $h = 1$:

x	0	1	2	3
$f(x)$	1	$\sqrt{2}$	3	$\sqrt{28}$

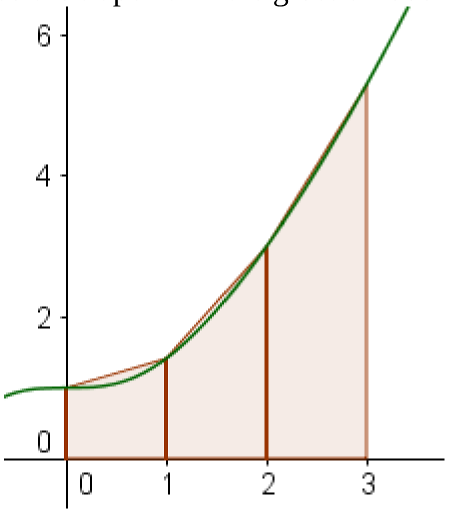
$$A \approx \tfrac{1}{2}\left[1+\sqrt{28}+2\left(\sqrt{2}+3\right)\right]=7.56$$

(b) With 6 strips of width $h = 0.5$:

x	0	0.5	1	1.5	2	2.5	3
y	1	$\sqrt{1.125}$	$\sqrt{2}$	$\sqrt{4.375}$	3	$\sqrt{16.625}$	$\sqrt{28}$

$$A \approx \tfrac{1}{4}\left[1+\sqrt{28}+2\left(\sqrt{1.125}+\sqrt{2}+\sqrt{4.375}+3+\sqrt{16.625}\right)\right]=7.39$$

(ii) From the graph of the function, it is clear that the trapezia lie above the curve, so the trapezium rule gives an over-estimate.



Notice that using more trapezia, in the example above, gives a lower value for the integral. You would expect using more trapezia to give an more accurate total, and this is consistent with the trapezium rule giving an overestimate in this case, as shown by the graph.

It is not always clear whether the trapezium rule gives an overestimate or an underestimate. In the example above, the curve is convex for the region being considered, and so the trapezia all lie above the curve. If the curve is always concave for the region being considered, the trapezia all lie below the curve. However if there is a point of inflection (so the curve changes between being convex and concave) then you cannot usually tell whether the trapezium rule gives an overestimate or an underestimate.

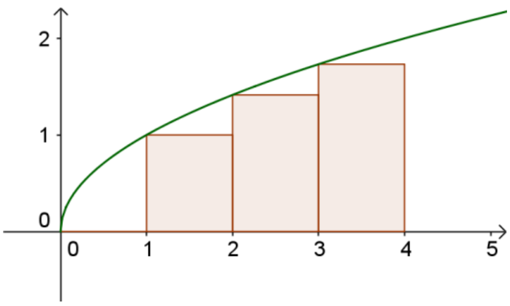
Upper and lower bounds for integrals using rectangles

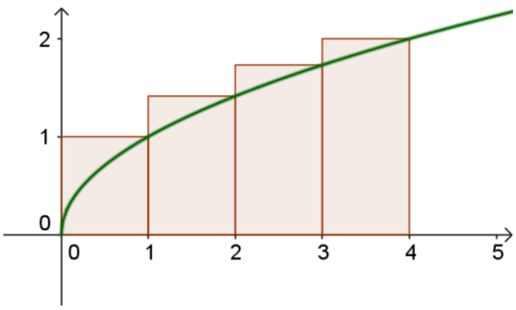
The trapezium rule gives you an approximate value for an integral, and you can often tell from the shape of the curve whether it is an overestimate or an underestimate, but you may not have much idea of how accurate the estimate is.

Using rectangles, rather than trapezia, can be useful in that if the curve does not have a turning point, rectangles that are either below the curve or above it can give a lower bound or an upper bound.

This is illustrated in the diagram below, which shows rectangles being used to approximate the area under the curve $y=\sqrt{x}$ between $x=0$ and $x=4$. (You can of course find this area exactly by integration – it works out to be $\frac{16}{3}$.)

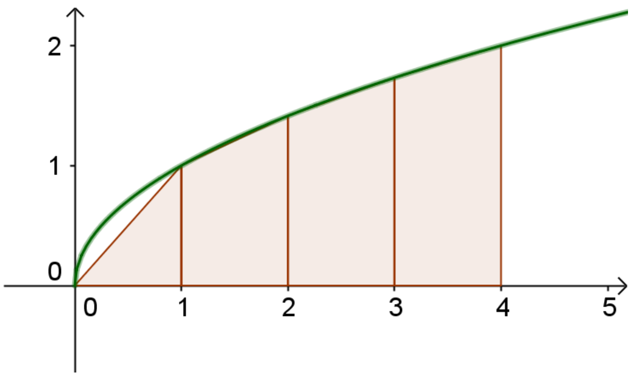
The diagram on the left shows rectangles that give a lower bound for the area (the first rectangle has height zero), and the diagram on the right shows rectangles that give an upper bound for the area.





In this illustration each of the rectangles has width 1.
The area on the left is given by $1\times\left(\sqrt{0}+\sqrt{1}+\sqrt{2}+\sqrt{3}\right)=4.146$
The area on the right is given by $1\times\left(\sqrt{1}+\sqrt{2}+\sqrt{3}+\sqrt{4}\right)=6.146$.
So the true area is between these two values.

In fact, since the curve is concave, the trapezium rule would give a better lower bound:



Area $=\frac{1}{2}\left(\sqrt{0}+2(\sqrt{1}+\sqrt{2}+\sqrt{3})+\sqrt{4}\right)=5.146$.

More generally, the total area of the rectangles for one bound is given by

$$h\left(\mathrm{f}(x_0)+\ldots+\mathrm{f}(x_{n-1})\right)$$

and the total area of the rectangles for the other upper bound is given by

$$h\left(\mathrm{f}(x_1)+\ldots+\mathrm{f}(x_n)\right)$$

where *h* is the width of the rectangles.

Which of these is the upper bound and which is the lower bound depends on the shape of the curve.