Section 1: Finding areas

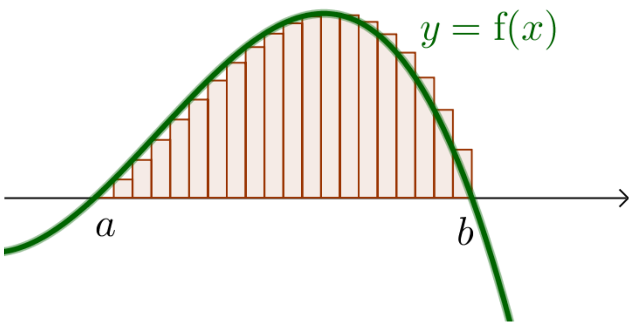
Notes and Examples

These notes contain subsections on

Reminder: the area between a curve and the x-axis

In AS (Year 1) you saw how integration can be used to find the area between a curve and the x-axis.

You can think of the area between a curve and the x-axis as the sum of a number of thin rectangles.



Each rectangle has height y (this varies depends on the value of x for each particular rectangle) and width δx (which is small).

The total area of the rectangles can be expressed as

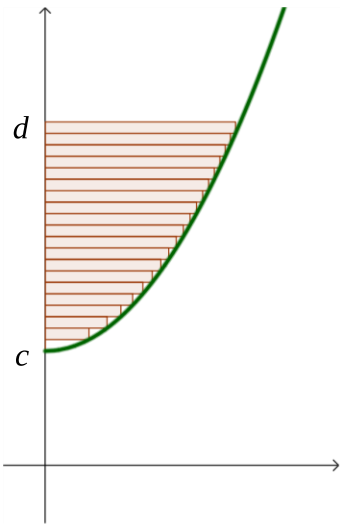
$$\sum y\delta x$$

In the limit, as $\delta x \to 0$, the sum becomes an integral.

$$\text{Area } = \int_a^b y\,\mathrm{d}x = \int_a^b \mathrm{f}(x)\,\mathrm{d}x$$

The area between a curve and the y-axis

The same idea can be used to find the area between a curve and the y-axis, but using horizontal strips instead.



However, in this case the strips have width x and height δy.

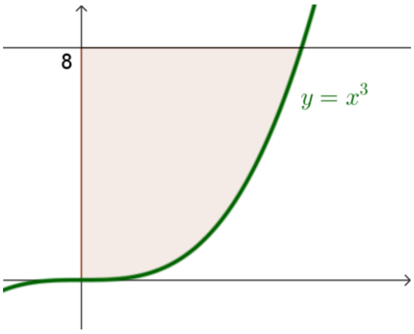
So the area is given by

$$\text{Area } = \int_c^d x \, \mathrm{d}y$$

Notice that the values of c and d are values of y rather than x, and in order to integrate with respect to y, you will need to express x in terms of y.

Example 1

Find the area of the region between the curve $y = x^3$, the y-axis and the line $y = 8$.



Solution

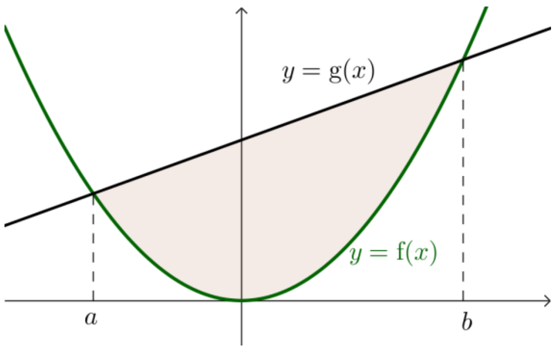
$y = x^3 \quad \Rightarrow x = y^{\frac{1}{3}}$

$$\begin{aligned}
\text{Area } &= \int_0^8 x \, \mathrm{d}y \\
&= \int_0^8 y^{\frac{1}{3}} \, \mathrm{d}y \\
&= \left[\tfrac{3}{4} y^{\frac{4}{3}} \right]_0^8 \\
&= \tfrac{3}{4}(8)^{\frac{4}{3}} - 0 \\
&= \tfrac{3}{4} \times 16 \\
&= 12
\end{aligned}$$

Edexcel A level Maths Integration 1 Notes & Examples

The area between a line and a curve

Sometimes you may need to find the area between a line and a curve, as shown in the diagram below.



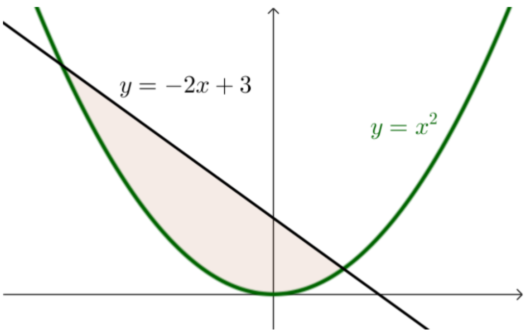
One way to approach this would be to find the area under the line, and the area under the curve, and subtract the second area from the first.

This could be written as $\int_a^b g(x)\,dx - \int_a^b f(x)\,dx$.

This integral can be rewritten as $\int_a^b \left(g(x) - f(x)\right)dx$. This is a more efficient way of calculating the area (in particular, it bypasses any issues about areas that are partly below the x-axis). This method is, in effect, dividing the area into vertical strips with heights $g(x) - f(x)$.

Example 2

Find the area of the region between the line $y = -2x + 3$ and the curve $y = x^2$ shown in the diagram below.



Solution

The first step is to find where the line and the curve intersect.

$$x^2 = -2x + 3$$
$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3 \text{ or } 1$$

$$\begin{aligned}\text{Area } &= \int_{-3}^{1}\left((-2x+3)-(x^2)\right)\mathrm{d}x\\ &= \int_{-3}^{1}\left(-2x+3-x^2\right)\mathrm{d}x\\ &= \left[-x^2+3x-\tfrac{1}{3}x^3\right]_{-3}^{1}\\ &= -1+3-\tfrac{1}{3}-(-9-9+9)\\ &= \tfrac{32}{3}\end{aligned}$$

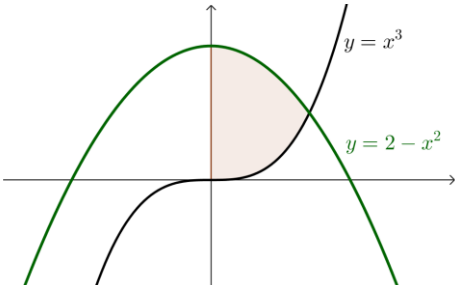
The area between two curves

The same approach can be used to find the area between two curves. It is important to sketch the curve, so that you can get an idea of the region involved, and roughly where the intersection points are, before you work out exactly where they are. You also need to know which curve you are going to subtract from which.

Example 3

Find the area of the region bounded by the curves $y = x^3$, $y = 2 - x^2$ and the y-axis.

Solution



At the intersection of the curves, $x^3 = 2 - x^2$

$$x^3 + x^2 - 2 = 0$$

By inspection, $x = 1$ is a root of the equation.

$$\begin{aligned}\text{Area } &= \int_{0}^{1}\left(2-x^2-x^3\right)\mathrm{d}x\\ &= \left[2x-\tfrac{1}{3}x^3-\tfrac{1}{4}x^4\right]_{0}^{1}\\ &= 2-\tfrac{1}{3}-\tfrac{1}{4}\\ &= \tfrac{17}{12}\end{aligned}$$

