# Section 3: Further techniques for integration

## Section test

1. $\int \frac{1}{x^2+2x}\,dx =$

(a) $\ln|x^2+2x|+c$

(b) $\frac{1}{2}\ln\left|\frac{x}{x+2}\right|+c$

(c) $\frac{\ln|x^2+2x|}{2x+2}+c$

(d) $\ln\left|\frac{x}{x+2}\right|+c$

2. $\int \frac{x^2+1}{x(x+1)^2}\,dx =$

(a) $\ln|x|+\frac{2}{x+1}+c$

(b) $\ln|x|+c$

(c) $\ln|x|-2\ln(x+1)^2+c$

(d) $2\ln|x|+\frac{1}{x+1}+c$

3. $\int \frac{x-2}{x^2(x+1)}\,dx =$

(a) $\ln|x(x+1)|-2\ln|x^2(x+1)|+c$

(b) $3\ln\left|\frac{x}{x+1}\right|-\frac{2}{x}+c$

(c) $3\ln\left|\frac{x}{x+1}\right|+\frac{2}{x}+c$

(d) $3\ln\left|\frac{x}{x+1}\right|-2\ln|x^2|+c$

4. Find the exact value of $\int_0^2 \frac{x+1}{(x+2)(x+3)}\,dx$.

5. Find the exact value of $\int_2^3 \frac{x^2}{(x-1)^2}\,dx$.

6. Find the exact value of $\int_0^1 \frac{4x+1}{(x+1)^2(x-2)}\,dx$.

7. Find the exact value of $\int_0^1 \frac{x}{2x+1}\,dx$.

8. Find the exact value of $\int_0^4 \frac{x+4}{\sqrt{4-x}}\,dx$.

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9. Find $\int_2^3 \frac{1}{x \ln x} \mathrm{d}x$

(a) 1

(b) $\ln \frac{3}{2}$

(c) $\ln\left(\frac{\ln 3}{\ln 2}\right)$

(d) 1.5

10. Find $\int_0^1 \frac{1-\mathrm{e}^{-3x}}{1+\mathrm{e}^{-3x}} \mathrm{d}x$

(a) $2\ln 2 - 1$

(b) $\frac{2}{3}\ln 2$

(c) $2\ln\left(\frac{1+\mathrm{e}^{-3}}{2}\right) - \mathrm{e}^{-3}$

(d) $1 + \frac{2}{3}\ln\left(\frac{1+\mathrm{e}^{-3}}{2}\right)$

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## Section test solutions

1. $\dfrac{1}{x^2+2x} = \dfrac{1}{x(x+2)}$

Putting into partial fractions:

$$\frac{1}{x(x+2)} \equiv \frac{A}{x} + \frac{B}{x+2}$$

$$1 \equiv A(x+2) + Bx$$

Let $x = 0$ $\quad 1 = 2A \Rightarrow A = \frac{1}{2}$

Let $x = -2$ $\quad 1 = -2B \Rightarrow B = -\frac{1}{2}$

$$\int \frac{1}{x^2+2x}\,dx = \int\left(\frac{1}{2x} - \frac{1}{2x+4}\right)dx$$

$$= \tfrac{1}{2}\ln|2x| - \tfrac{1}{2}\ln|2x+4| + c$$

$$= \tfrac{1}{2}\ln\left|\frac{2x}{2x+4}\right| + c$$

$$= \tfrac{1}{2}\ln\left|\frac{x}{x+2}\right| + c$$

2. Putting into partial fractions:

$$\frac{x^2+1}{x(x+1)^2} \equiv \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$x^2+1 \equiv A(x+1)^2 + Bx(x+1) + Cx$$

Let $x = 0$ $\quad 1 = A$

Let $x = -1$ $\quad 2 = -C \Rightarrow C = -2$

Equating coefficients of $x$: $\quad 0 = 2A + B + C$

$$= 2 + B - 2$$

$$B = 0$$

$$\int \frac{x^2+1}{x(x+1)^2}\,dx = \int\left(\frac{1}{x} - \frac{2}{(x+1)^2}\right)dx$$

$$= \ln|x| + \frac{2}{x+1} + c$$

3. $\dfrac{x-2}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

$$x-2 = Ax(x+1) + B(x+1) + Cx^2$$

Let $x = 0$ $\quad \Rightarrow -2 = B$

Let $x = -1$ $\quad \Rightarrow -3 = C$

Equating coefficients of $x^2$: $\quad 0 = A + C \quad \Rightarrow A = 3$

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$$\int \frac{x-2}{x^2(x+1)}dx = \int\left(\frac{3}{x}-\frac{2}{x^2}-\frac{3}{x+1}\right)dx$$
$$= 3\ln|x| + \frac{2}{x} - 3\ln|x+1| + c$$
$$= 3\ln\left|\frac{x}{x+1}\right| + \frac{2}{x} + c$$

4. $\dfrac{x+1}{(x+2)(x+3)} = \dfrac{A}{x+2} + \dfrac{B}{x+3}$

$x+1 = A(x+3) + B(x+2)$

Putting $x = -2$: $\quad -1 = A$

Putting $x = -3$: $\quad -2 = -B \Rightarrow B = 2$

$$\int_0^2 \frac{x+1}{(x+2)(x+3)}dx = \int_0^2\left(\frac{2}{x+3}-\frac{1}{x+2}\right)dx$$
$$= \left[2\ln|x+3| - \ln|x+2|\right]_0^2$$
$$= 2\ln 5 - \ln 4 - 2\ln 3 + \ln 2$$
$$= \ln\left(\frac{5^2\times 2}{4\times 3^2}\right)$$
$$= \ln\tfrac{25}{18}$$

5. Let $u = x-1 \quad \Rightarrow \dfrac{du}{dx} = 1 \quad \Rightarrow dx = du$

When $x = 2, u = 1$

When $x = 3, u = 2$

$$\int_2^3 \frac{x^2}{(x-1)^2}dx = \int_1^2 \frac{(u+1)^2}{u^2}du$$
$$= \int_1^2 \frac{u^2+2u+1}{u^2}du$$
$$= \int_1^2\left(1+\frac{2}{u}+\frac{1}{u^2}\right)du$$
$$= \left[u + 2\ln u - \frac{1}{u}\right]_1^2$$
$$= 2 + 2\ln 2 - \tfrac{1}{2} - 1 - 0 + 1$$
$$= \tfrac{3}{2} + 2\ln 2$$

6. $\dfrac{4x+1}{(x+1)^2(x-2)} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-2}$

$4x+1 = A(x+1)(x-2) + B(x-2) + C(x+1)^2$

Let $x = 2 \quad \Rightarrow 9 = 9C \quad \Rightarrow C = 1$

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Let $x=-1 \Rightarrow -3=-3B \Rightarrow B=1$

Equating coefficients of $x^2 \Rightarrow 0=A+C \Rightarrow A=-1$

$$\int_0^1 \frac{4x+1}{(x+1)^2(x-2)}dx = \int_0^1 \left(-\frac{1}{x+1}+\frac{1}{(x+1)^2}+\frac{1}{x-2}\right)dx$$

$$= \left[-\ln|x+1|-\frac{1}{x+1}+\ln|x-2|\right]_0^1$$

$$= -\ln 2-\tfrac{1}{2}+\ln 1-(-\ln 1-1+\ln 2)$$

$$= \tfrac{1}{2}-2\ln 2$$

7. Let $u=2x+1 \Rightarrow \frac{du}{dx}=2 \Rightarrow dx=\frac{1}{2}du$

When $x=0, u=1$

When $x=1, u=3$

Also, $u=2x+1 \Rightarrow x=\frac{1}{2}(u-1)$

$$\int_0^1 \frac{x}{2x+1}\,dx = \int_1^3 \frac{\frac{1}{2}(u-1)}{u}\times\frac{1}{2}du$$

$$= \tfrac{1}{4}\int_1^3 \left(1-u^{-1}\right)du$$

$$= \tfrac{1}{4}\left[u-\ln u\right]_1^3$$

$$= \tfrac{1}{4}\left[(3-\ln 3)-(1-\ln 1)\right]$$

$$= \tfrac{1}{2}-\tfrac{1}{4}\ln 3$$

8. Let $u=4-x \Rightarrow \frac{du}{dx}=-1 \Rightarrow dx=-du$

Also $x=4-u$

When $x=0, u=4$

$x=4, u=0$

$$\int_0^4 \frac{x+4}{\sqrt{4-x}}dx = -1\int_4^0 (4-u+4)u^{-\frac{1}{2}}\,du$$

$$= -1\int_4^0 u^{-\frac{1}{2}}(8-u)\,du$$

$$= -1\int_4^0 \left(8u^{-\frac{1}{2}}-u^{\frac{1}{2}}\right)du$$

$$= -1\left[16u^{\frac{1}{2}}-\tfrac{2}{3}u^{\frac{3}{2}}\right]_4^0$$

$$= -1\left[0-16\sqrt{4}+\tfrac{2}{3}\sqrt{4^3}\right]$$

$$= \tfrac{80}{3}$$

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9. $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

When $x = 2, u = \ln 2$

When $x = 3, u = \ln 3$

$$\int_2^3 \frac{1}{x \ln x} dx = \int_{\ln 2}^{\ln 3} \frac{1}{xu} \times x\,du$$

$$= \int_{\ln 2}^{\ln 3} \frac{1}{u} du$$

$$= [\ln u]_{\ln 2}^{\ln 3}$$

$$= \ln(\ln 3) - \ln(\ln 2)$$

$$= \ln\left(\frac{\ln 3}{\ln 2}\right)$$

10. Let $u = e^{-3x} \quad \Rightarrow \frac{du}{dx} = -3e^{-3x} \quad \Rightarrow dx = -\frac{1}{3e^{-3x}} du$

$$\int \frac{1 - e^{-3x}}{1 + e^{-3x}} dx = \int \frac{1-u}{1+u} \times -\frac{1}{3u} du$$

$$= \int -\frac{1-u}{3u(1+u)} du$$

$$= \int \frac{u-1}{3u(1+u)} du$$

$$\frac{u-1}{3u(u+1)} = \frac{A}{3u} + \frac{B}{u+1}$$

$$u - 1 = A(u+1) + 3Bu$$

Let $u = 0 \quad \Rightarrow A = -1$

Let $u = -1 \quad \Rightarrow B = \frac{2}{3}$

$$\int \frac{u-1}{3u(1+u)} du = \int \left( \frac{2}{3(u+1)} - \frac{1}{3u} \right) du$$

$$= \tfrac{2}{3}\ln|u+1| - \tfrac{1}{3}\ln u$$

$$= \tfrac{2}{3}\ln\left|e^{-3x}+1\right| - \tfrac{1}{3}\ln\left|e^{-3x}\right|$$

$$= \tfrac{2}{3}\ln\left|e^{-3x}+1\right| - \tfrac{1}{3} \times -3x$$

$$= \tfrac{2}{3}\ln\left|e^{-3x}+1\right| + x$$

$$\left[\tfrac{2}{3}\ln\left|e^{-3x}+1\right| + x\right]_0^1 = \tfrac{2}{3}\ln\left|e^{-3}+1\right| + 1 - \tfrac{2}{3}\ln 2$$

$$= 1 + \tfrac{2}{3}\ln\left(\frac{1+e^{-3}}{2}\right)$$