Section 3: Further techniques for integration

Notes and Examples

These notes contain subsections on

- **Using partial fractions in integration**
- **Further integration by substitution**

Using partial fractions in integration

The technique of splitting a fraction into partial fractions can be used to integrate some functions involving fractions, as the following two examples illustrate.

Example 1

Find $\int \frac{1}{x^2+2x-3}\,\mathrm{d}x$

Solution

The integrand splits into two partial fractions, like this:

$$\frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)} = \frac{A}{x+3} + \frac{B}{x-1}$$

Multiplying through by $(x + 3)(x - 1)$: $1 = A(x - 1) + B(x + 3)$

Substituting $x = 1$: $1 = 4B \Rightarrow B = \frac{1}{4}$

Substituting $x = -3$: $1 = -4A \Rightarrow A = -\frac{1}{4}$

So $$\frac{1}{x^2+2x-3} = \frac{-1}{4(x+3)} + \frac{1}{4(x-1)}$$

$$\begin{aligned}\int \frac{1}{x^2+2x-3}\,\mathrm{d}x &= \int\left(-\frac{1}{4(x+3)} + \frac{1}{4(x-1)}\right)\mathrm{d}x \\ &= \frac{1}{4}\int\left(\frac{1}{x-1} - \frac{1}{x+3}\right)\mathrm{d}x \\ &= \tfrac{1}{4}\ln|x-1| - \tfrac{1}{4}\ln|x+3| + c \\ &= \frac{1}{4}\ln\left|\frac{x-1}{x+3}\right| + c\end{aligned}$$

Example 2

Evaluate $\int_1^2 \frac{x+1}{x^3+2x^2}\,\mathrm{d}x$, giving your answer in exact form.

Solution

$$\frac{x+1}{x^3+2x^2} = \frac{x+1}{x^2(x+2)} = \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x+2}$$

Multiplying through by $x^2(x+2)$: $x + 1 = A(x + 2) + Bx(x + 2) + Cx^2$

Substituting $x = -2$: $-1 = 4C \Rightarrow C = -\frac{1}{4}$

Substituting $x = 0$: $1 = 2A \Rightarrow A = \frac{1}{2}$

Equating coefficients of x^2: $0 = B + C \Rightarrow B = \frac{1}{4}$

So $\frac{x+1}{x^3+2x^2} = \frac{1}{2x^2} + \frac{1}{4x} - \frac{1}{4(x+2)}$

$$\int_1^2 \frac{x+1}{x^3+2x^2}\,\mathrm{d}x = \frac{1}{4}\int_1^2 \left(\frac{2}{x^2} + \frac{1}{x} - \frac{1}{x+2}\right)\mathrm{d}x$$

$$= \frac{1}{4}\left[-\frac{2}{x} + \ln x - \ln(x+2)\right]_1^2$$

$$= \tfrac{1}{4}\left(-1 + \ln 2 - \ln 4 + 2 - \ln 1 + \ln 3\right)$$

$$= \tfrac{1}{4}\left(1 + \ln 2 - \ln 4 + \ln 3\right)$$

$$= \tfrac{1}{4}\left(1 + \ln \tfrac{3}{2}\right)$$

Further integration by substitution

The previous section covered the method of integration by substitution. All the integrations that were covered in that section were of the form $\int \mathrm{f}'(x)[\mathrm{g}(\mathrm{f}(x)]\mathrm{d}x$. In cases like these, it is easy to spot that the substitution $u = \mathrm{f}(x)$ can be used, e.g. to integrate $\sqrt{3x-1}$ you could use $u = 3x - 1$, to integrate $x\sin(x^2+1)$ you can use $u = x^2 + 1$, or to integrate $\cos x\,\mathrm{e}^{\sin x}$ you can use $u = \sin x$. All of these integrations can also be done by inspection.

However, some integrations which cannot be done by inspection can be done by substitution, as in Example 3 below.

Example 3

Find $\int x\sqrt{1+2x}\,\mathrm{d}x$.

Solution

Let $u = 1 + 2x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2$

$\Rightarrow \mathrm{d}u = 2\mathrm{d}x$

$\Rightarrow \mathrm{d}x = \frac{1}{2}\mathrm{d}u$

Also $x = \frac{1}{2}(u-1)$

$$\int x\sqrt{1+2x}\,dx = \int \tfrac{1}{2}(u-1)\sqrt{u} \times \tfrac{1}{2}\,du$$
$$= \tfrac{1}{4}\int (u^{3/2} - u^{1/2})\,du$$
$$= \tfrac{1}{4}\left[\tfrac{2}{5}u^{\frac{5}{2}} - \tfrac{2}{3}u^{\frac{3}{2}}\right] + c$$
$$= \tfrac{1}{10}(1+2x)^{5/2} - \tfrac{1}{6}(1+2x)^{3/2} + c$$

It's a good idea to take out this fraction in front of the integral to simplify the integration.

Notice that when the integration is indefinite, as in Example 3, you must express the final answer in terms of x, not the new variable u.