Section 4: Integration by parts

Section test

1. The formula for integration by parts is

(a) $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv + \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$ (b) $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$

(c) $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = v\frac{\mathrm{d}u}{\mathrm{d}x} + \int uv\ \mathrm{d}x$ (d) $\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = v\frac{\mathrm{d}u}{\mathrm{d}x} - \int uv\ \mathrm{d}x$

2. $\int x\mathrm{e}^{-x}\mathrm{d}x =$

(a) $x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$ (b) $-x\mathrm{e}^{-x} - \mathrm{e}^{-x} + c$

(c) $-x\mathrm{e}^{-x} + \mathrm{e}^{-x} + c$ (d) $x\mathrm{e}^{-x} + \mathrm{e}^{-x} + c$

3. $\int x\cos 2x\ \mathrm{d}x =$

(a) $2x\sin 2x + 4\cos 2x + c$ (b) $\frac{1}{2}x\sin 2x - \frac{1}{4}\cos 2x + c$

(c) $x\sin 2x + \cos 2x + c$ (d) $\frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + c$

4. $\int (2x+1)\ln x\ \mathrm{d}x =$

(a) $\left(x^2+x\right)\ln x - \frac{1}{2}x^2 - x + c$ (b) $\left(x^2+x\right)\ln x + \frac{1}{2}x^2 + x + c$

(c) $\left(2x+1\right)\ln x - \frac{1}{2}x^2 - x + c$ (d) $\left(2x+1\right)\ln x - x - 1 + c$

5. Find the exact value of $\int_0^2 (1-x)\mathrm{e}^{2x}\mathrm{d}x$.

6. Find the exact value of $\int_1^2 x^3 \ln x\ \mathrm{d}x$.

7. Find the exact value of $\int_{-\pi/2}^{\pi/2} x\cos x\ \mathrm{d}x$.

8. Find the exact value of $\int_0^1 x^2\mathrm{e}^{-x}\ \mathrm{d}x$

9. Find the exact value of the area under the graph of $y = \ln x$ between $x = 1$ and $x = 5$.

10. The area between the graph $y = x \sin x$ and the x axis between $x = 0$ and $x = 2\pi$ is

(a) 2π (b) 4π

(c) 1 (d) 2

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Section test solutions

1. By definition, the formula for integration by parts is
$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

2. Let $u = x \Rightarrow \frac{du}{dx} = 1$
$$\frac{dv}{dx} = e^{-x} \Rightarrow v = -e^{-x}$$
Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$:
$$\int xe^{-x}dx = x\left(-e^{-x}\right) - \int\left(-e^{-x}\right)(1)\ dx$$
$$= -xe^{-x} - e^{-x} + c$$

3. Let $u = x \Rightarrow \frac{du}{dx} = 1$
$$\frac{dv}{dx} = \cos 2x \Rightarrow v = \tfrac{1}{2}\sin 2x$$
Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$:
$$\int x\cos 2x\ dx = x\left(\tfrac{1}{2}\sin 2x\right) - \int\left(\tfrac{1}{2}\sin 2x\right)\times 1\ dx$$
$$= \tfrac{1}{2}x\sin 2x - \left(-\tfrac{1}{4}\cos 2x\right) + c$$
$$= \tfrac{1}{2}x\sin 2x + \tfrac{1}{4}\cos 2x + c$$

4. Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$
$$\frac{dv}{dx} = 2x + 1 \Rightarrow v = x^2 + x$$
Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$
$$\int (2x+1)\ln x\ dx = (\ln x)\left(x^2 + x\right) - \int\left(x^2 + x\right)\left(\frac{1}{x}\right)\ dx$$
$$= \left(x^2 + x\right)\ln x - \int x + 1\ dx$$
$$= \left(x^2 + x\right)\ln x - \left(\tfrac{1}{2}x^2 + x\right) + c$$
$$= \left(x^2 + x\right)\ln x - \tfrac{1}{2}x^2 - x + c$$

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5. Let $u = 1 - x \Rightarrow \frac{du}{dx} = -1$

$\frac{dv}{dx} = e^{2x} \Rightarrow v = \frac{1}{2}e^{2x}$

Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$:

$$\int_0^2 (1-x)e^{2x}dx = \left[(1-x)\left(\tfrac{1}{2}e^{2x}\right)\right]_0^2 - \int_0^2 \left(\tfrac{1}{2}e^{2x} \times -1\right)dx$$

$$= \left[(1-x)\left(\tfrac{1}{2}e^{2x}\right) + \tfrac{1}{4}e^{2x}\right]_0^2$$

$$= \left[\tfrac{1}{4}(2-2x+1)e^{2x}\right]_0^2$$

$$= \left[\tfrac{1}{4}(3-2x)e^{2x}\right]_0^2$$

$$= -\tfrac{1}{4}e^4 - \tfrac{3}{4}$$

6. Let $u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = x^3 \Rightarrow v = \frac{1}{4}x^4$

Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$:

$$\int_1^2 x^3 \ln x dx = \left[\ln x\left(\tfrac{1}{4}x^4\right)\right]_1^2 - \int_1^2 \left(\tfrac{1}{4}x^4\right)\left(\frac{1}{x}\right)dx$$

$$= \left[\tfrac{1}{4}x^4 \ln x\right]_1^2 - \int_1^2 \tfrac{1}{4}x^3\ dx$$

$$= \left[\tfrac{1}{4}x^4 \ln x - \tfrac{1}{16}x^4\right]_1^2$$

$$= \left[\tfrac{1}{16}x^4(4\ln x - 1)\right]_1^2$$

$$= \tfrac{1}{16} \times 2^4(4\ln 2 - 1) - \tfrac{1}{16} \times 1^4(4\ln 1 - 1)$$

$$= 4\ln 2 - 1 + \tfrac{1}{16}$$

$$= 4\ln 2 - \tfrac{15}{16}$$

7. Let $u = x \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \cos x \Rightarrow v = \sin x$

Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\cos x \, dx = \left[x\sin x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} - \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin x \, dx$$
$$= \left[x\sin x + \cos x\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$
$$= \left[\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right] - \left[-\tfrac{\pi}{2}\sin\left(-\tfrac{\pi}{2}\right) + \cos\left(-\tfrac{\pi}{2}\right)\right]$$
$$= \left[\tfrac{\pi}{2} + 0\right] - \left[\tfrac{\pi}{2} + 0\right]$$
$$= 0$$

8. $\int_0^1 x^2 e^{-x} \, dx$

Let $u = x^2 \quad \Rightarrow \frac{du}{dx} = 2x$

Let $\frac{dv}{dx} = e^{-x} \quad \Rightarrow v = -e^{-x}$

Using: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int_0^1 x^2 e^{-x} \, dx = \left[-x^2 e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}dx$$

Now using integration by parts on $\int_0^1 2xe^{-x}dx$

Let $u = 2x \quad \Rightarrow \frac{du}{dx} = 2$

Let $\frac{dv}{dx} = e^{-x} \quad \Rightarrow v = -e^{-x}$

$$\int_0^1 2xe^{-x}dx = \left[-2xe^{-x}\right]_0^1 + \int_0^1 2e^{-x}dx$$
$$= \left[-2xe^{-x} - 2e^{-x}\right]_0^1$$

So $\int_0^1 x^2 e^{-x} \, dx = \left[-x^2 e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1$
$$= -e^{-1} - 2e^{-1} - 2e^{-1} + 2$$
$$= 2 - \frac{5}{e}$$

9. The integral you require to find this area is $\int_1^5 \ln x \, dx$

This can be alternatively written as $\int_1^5 1 \times \ln x \, dx$

You can now use parts to do this integral.

Let $\quad u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\frac{dv}{dx} = 1 \quad \Rightarrow v = x$

Using: $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int_1^5 1\ln x\ dx = [\ln x(x)]_1^5 - \int_1^5 x\left(\frac{1}{x}\right)\ dx$$
$$= [x\ln x]_1^5 - \int_1^5 1\ dx$$
$$= [x\ln x - x]_1^5$$
$$= (5\ln 5 - 5) - (\ln 1 - 1)$$
$$= 5\ln 5 - 4$$

10. The area between $x = 0$ and $x = \pi$ is above the x-axis, and the area between $x = \pi$ and $x = 2\pi$ is below the x-axis.
You can split the integral into two separate ones, and add their absolute values to avoid this problem. Firstly, find $\int_0^\pi x\sin x\ dx$:

Let $u = x \quad \Rightarrow \frac{du}{dx} = 1$

$\frac{dv}{dx} = \sin x \Rightarrow \quad v = -\cos x$

Using $\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$

$$\int_0^\pi x\sin x\ dx = [x(-\cos x)]_0^\pi - \int_0^\pi (-\cos x)(1)dx$$
$$= [-x\cos x + \sin x]_0^\pi$$
$$= (-\pi\cos\pi + \sin\pi) - (\sin 0)$$
$$= \pi + 0 - 0$$
$$= \pi$$

The area between $x = 0$ and $x = \pi$ is π.

The area between $x = \pi$ and $x = 2\pi$ is given by

$$[-x\cos x + \sin x]_\pi^{2\pi} = (-2\pi\cos 2\pi + \sin 2\pi) - (-\pi\cos\pi + \sin\pi)$$
$$= -2\pi + 0 - \pi - 0$$
$$= -3\pi$$

The area between $x = \pi$ and $x = 2\pi$ is -3π

The total area between the graph and the x-axis is therefore 4π.