Section 4: Integration by parts

Notes and Examples

These notes contain subsections on

Integration by parts

Integration by parts is another technique which can sometimes be used to integrate the product of two simpler functions. It is useful in many cases where a substitution will not help, although it cannot be used for all functions.

Suppose you want to integrate $x\cos x$. This is the product of two functions which we can integrate, x and $\cos x$. This suggests that reversing the product rule might give us a method.

Try differentiating $x\sin x$ using the product rule:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}(x\sin x) &= x\times\cos x+\sin x\times 1\\ &= x\cos x+\sin x\end{aligned}$$

So $\qquad \int(x\cos x+\sin x)\mathrm{d}x = x\sin x+c$

and $\qquad \int x\cos x\ \mathrm{d}x+\int\sin x\ \mathrm{d}x = x\sin x+c$

and finally $\qquad \int x\cos x\ \mathrm{d}x = x\sin x-\int\sin x\ \mathrm{d}x+c$

$$= x\sin x+\cos x+c$$

We need to take the cleverness out of this method and make it more systematic!

Starting with the product rule for differentiation:

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x}+v\frac{\mathrm{d}u}{\mathrm{d}x}$$

$$\Rightarrow \qquad u\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}(uv)-v\frac{\mathrm{d}u}{\mathrm{d}x}$$

Now integrate both sides with respect to x:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = \int\frac{\mathrm{d}}{\mathrm{d}x}(uv)\mathrm{d}x - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Differentiating *uv*, then integrating the result, just leaves *uv*!

$\Rightarrow$ $$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

This formula is called *integration by parts*.

This formula can be used to find the integral of $x \cos x$ shown earlier:

Split the integrand $x \cos x$ into two parts u and $\frac{dv}{dx}$:

$$u = x \text{ , } \frac{dv}{dx} = \cos x \Rightarrow v = \int \cos x \, dx = \sin x$$

You don't need a '+c' here, as it is added to the final result

So $\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$

$\Rightarrow$ $$\int x \cos x \, dx = x \sin x - \int \sin x \frac{d}{dx}(x) dx$$
$$= x \sin x - \int \sin x \, dx$$
$$= x \sin x + \cos x + c$$

The choice of how to divide up the integrand between u and $\frac{dv}{dx}$ is a matter of experience. Usually, u is a simple function, such as a linear function of x, which becomes even simpler when differentiated.

However, when the integrand involves a logarithm, this has to be 'u': $\ln x$ can't be integrated easily, so it can't be $\frac{dv}{dx}$. This is shown in the following example:

Example 1

Find $\int x \ln x dx$.

Solution

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$
$$\frac{dv}{dx} = x \Rightarrow v = \tfrac{1}{2} x^2$$

Using the formula for integration by parts:

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

$\Rightarrow$ $$\int x \ln x \, dx = \tfrac{1}{2} x^2 \ln x - \int \tfrac{1}{2} x^2 \times \frac{1}{x} dx$$
$$= \tfrac{1}{2} x^2 \ln x - \int \tfrac{1}{2} x \, dx$$
$$= \tfrac{1}{2} x^2 \ln x - \tfrac{1}{4} x^2 + c$$

Using integration by parts more than once

Sometimes you will need to use integration by parts more than once, as in the following example.

Example 2

Find $\int (x+1)^2 \mathrm{e}^{-2x}\,\mathrm{d}x$

Solution

$$u = (x+1)^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2(x+1)$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-2x} \Rightarrow v = \int \mathrm{e}^{-2x}\,\mathrm{d}x = -\tfrac{1}{2}\mathrm{e}^{2x}$$

Using the formula for integration by parts:

$$\int u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = uv - \int v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

$$\int (x+1)^2 \mathrm{e}^{-2x}\,\mathrm{d}x = (x+1)^2 \times -\tfrac{1}{2}\mathrm{e}^{-2x} - \int 2(x+1)\times -\tfrac{1}{2}\mathrm{e}^{-2x}\mathrm{d}x$$

$$= -\tfrac{1}{2}(x+1)^2\mathrm{e}^{-2x} + \int (x+1)\,\mathrm{e}^{-2x}\mathrm{d}x$$

Be careful with signs

The new integral we need to find, $\int (x+1)\,\mathrm{e}^{-2x}\mathrm{d}x$, also has to be done using integration by parts.

$$u = x+1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 1$$

$$\frac{\mathrm{d}v}{\mathrm{d}x} = \mathrm{e}^{-2x} \Rightarrow v = \int \mathrm{e}^{-2x}\,\mathrm{d}x = -\tfrac{1}{2}\mathrm{e}^{2x}$$

Using the formula for integration by parts:

$$\int (x+1)\mathrm{e}^{-2x}\,\mathrm{d}x = (x+1)\times -\tfrac{1}{2}\mathrm{e}^{-2x} - \int 1\times -\tfrac{1}{2}\mathrm{e}^{-2x}\mathrm{d}x$$

$$= -\tfrac{1}{2}(x+1)\mathrm{e}^{-2x} + \int \tfrac{1}{2}\mathrm{e}^{-2x}\mathrm{d}x$$

$$= -\tfrac{1}{2}(x+1)\mathrm{e}^{-2x} + \tfrac{1}{2}\times -\tfrac{1}{2}\mathrm{e}^{-2x} + c$$

$$= -\tfrac{1}{2}(x+1)\mathrm{e}^{-2x} - \tfrac{1}{4}\mathrm{e}^{-2x} + c$$

Again, be careful with signs

So $\int (x+1)^2\mathrm{e}^{-2x}\,\mathrm{d}x = -\tfrac{1}{2}(x+1)^2\mathrm{e}^{-2x} - \tfrac{1}{2}(x+1)\mathrm{e}^{-2x} - \tfrac{1}{4}\mathrm{e}^{-2x} + c$

Definite integration by parts

When using integration by parts on a definite integral, the formula for integration by parts becomes

$$\int_a^b u\frac{\mathrm{d}v}{\mathrm{d}x}\mathrm{d}x = [uv]_a^b - \int_a^b v\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{d}x$$

Notice that the 'uv' part of the formula should be evaluated between the limits, as in this final example:

Example 3

Find $\int_0^{\pi/6} x\sin 2x \; dx$.

Solution

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$\frac{dv}{dx} = \sin 2x \Rightarrow v = \int \sin 2x \; dx = -\tfrac{1}{2}\cos 2x$$

Using the formula for integration by parts:

$$\int u\frac{dv}{dx}dx = uv - \int v\frac{du}{dx}dx$$

$$\int_0^{\pi/6} x\sin 2x \; dx = \left[-\tfrac{1}{2}x\cos 2x\right]_0^{\pi/6} - \int(-\tfrac{1}{2}\cos 2x).1dx$$

$$= \left(-\tfrac{1}{2}\times\tfrac{\pi}{6}\cos\tfrac{\pi}{3} + \tfrac{1}{2}\times 0\times\cos 0\right) + \int \tfrac{1}{2}\cos 2x dx$$

$$= -\tfrac{\pi}{24} + \left[\tfrac{1}{4}\sin 2x\right]_0^{\pi/6}$$

$$= -\tfrac{\pi}{24} + \tfrac{1}{4}\sin\tfrac{\pi}{3} - \tfrac{1}{4}\sin 0$$

$$= -\frac{\pi}{24} + \frac{\sqrt{3}}{8}$$

$$= \frac{3\sqrt{3} - \pi}{24}$$

Remember that $\cos\tfrac{\pi}{3} = \tfrac{1}{2}$ and $\sin\tfrac{\pi}{3} = \tfrac{1}{2}\sqrt{3}$

