Section 2: Integration by substitution

Notes and Examples

These notes contain subsections on

- Integration by substitution
- Integrating exponential functions
- Logarithmic integrals
- Integrating trigonometric functions
- Using trigonometric identities in integration
- Using standard patterns

Integration by substitution

The chain rule allows you to differentiate a function of x by making a substitution of another variable u, say. What is the corresponding integration method?

Suppose you want to find the integral of $(2x + 1)^5$. If you wanted to differentiate this function, you could use the substitution $u = 2x + 1$ and then apply the chain rule. However, you are trying to integrate with respect to x, and if you integrate u^5 to give $\frac{1}{6}u^6$, then you have integrated with respect to u. The key is the '$\mathrm{d}x$' in the integral – this needs to be changed to $\mathrm{d}u$, using the equation

$$\mathrm{d}x = \frac{\mathrm{d}x}{\mathrm{d}u}\mathrm{d}u$$

which comes from letting $\delta x \to 0$ in the equation

$$\delta x \approx \frac{\mathrm{d}x}{\mathrm{d}u}\delta u\ .$$

Example 1 shows how the method is set out.

Example 1

Integrate $\int (2x+1)^5 \mathrm{d}x$

Solution

Let $u = 2x+1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2 \quad \Rightarrow \mathrm{d}u = 2\,\mathrm{d}x \quad \Rightarrow \mathrm{d}x = \frac{1}{2}\mathrm{d}u$

$$\Rightarrow \quad \int (2x+1)^5 \mathrm{d}x = \int u^5 \times \tfrac{1}{2}\mathrm{d}u$$
$$= \frac{1}{12}u^6 + c$$
$$= \frac{1}{12}(2x+1)^6 + c$$

In the case of Example 1, the function to be integrated was of the form $y=(ax+b)^n$. Functions like these can alternatively be integrated 'by inspection', by thinking about reversing the process of differentiating.

In general, $y=(ax+b)^n \quad \Rightarrow \quad \int y\,\mathrm{d}x=\frac{1}{a}\times\frac{1}{n+1}(ax+b)^{n+1}+c$

The derivative of $(ax + b)$ is a, so you need to divide by a

The integral of $(\ldots)^n$ is $\frac{1}{n+1}(\ldots)^{n+1}$

You should always check your integration by differentiating the result (you may be able to do this in your head once you have had some practice).

Example 2

Find the following indefinite integrals.

(i) $\int(5-3x)^6\,\mathrm{d}x$

(ii) $\int\frac{1}{(2x-3)^3}\,\mathrm{d}x$

(iii) $\int\frac{1}{\sqrt[3]{4x-1}}\,\mathrm{d}x$

Solution

(i) $\int(5-3x)^6\,\mathrm{d}x=-\frac{1}{3}\times\frac{1}{7}(5-3x)^7+c$

$=-\frac{1}{21}(5-3x)^7+c$

Check: differentiating this result gives $-\frac{1}{21}\times-3\times7(5-3x)^6=(5-3x)^6$

(ii) $\int\frac{1}{(2x-3)^3}\,\mathrm{d}x=\int(2x-3)^{-3}\,\mathrm{d}x$

$=\frac{1}{2}\times-\frac{1}{2}(2x-3)^{-2}+c$

$=-\frac{1}{4(2x-3)^2}+c$

Check: differentiating this result gives $-\frac{1}{4}\times2\times-2(2x+3)^{-3}=(2x+3)^{-3}$

(iii) $\int\frac{1}{\sqrt[3]{4x-1}}\,\mathrm{d}x=\int(4x-1)^{-\frac{1}{3}}\,\mathrm{d}x$

$=\frac{1}{4}\times\frac{3}{2}(4x-1)^{\frac{2}{3}}+c$

$=\frac{3}{8}(4x-1)^{\frac{2}{3}}+c$

Check: differentiating this result gives $\frac{3}{8}\times4\times\frac{2}{3}(4x-1)^{-\frac{1}{3}}=(4x-1)^{-\frac{1}{3}}$

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Integrating exponential functions

Remember that the derivative of e^x is e^x. Therefore $\int e^x \, dx = e^x + c$.

Similarly, since the derivative of e^{kx} is ke^{kx}: $\int e^{kx} \, dx = \frac{1}{k} e^{kx} + c$.

Example 3

Evaluate $\int_0^2 \sqrt{e^x} \, dx$, giving your answer in terms of e.

Solution

$$\begin{aligned}\int_0^2 \sqrt{e^x} \, dx &= \int_0^2 e^{\frac{1}{2}x} \, dx \\ &= \left[2e^{\frac{1}{2}x} \right]_0^2 \\ &= 2e^1 - 2e^0 \\ &= 2e - 2\end{aligned}$$

Logarithmic integrals

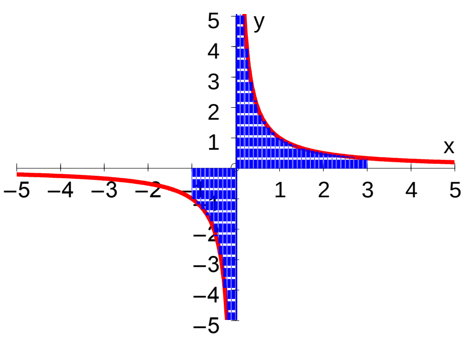
The derivative of $\ln x$ is $\frac{1}{x}$. It follows that $\int \frac{1}{x} \, dx = \ln x + c$, provided $x > 0$. Also, if $x < 0$, the derivative of $\ln(-x)$ is $\frac{1}{(-x)}.(-1) = \frac{1}{x}$, so $\int \frac{1}{x} \, dx = \ln(-x) + c$. These two results are sometimes combined using a modulus sign:

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

When dealing with an indefinite integral, you need to include the modulus signs, unless a restricted domain is given so that only positive values of x are involved.

Care needs to be taken when using this result. The domain of x must be either $x > 0$ or $x < 0$ – you cannot integrate across zero.

For example, suppose you wanted to find $\int_{-1}^3 \frac{1}{x} \, dx$. The graph below illustrates the area represented by this integral.



From the graph, you can see that this area is not defined, as it includes the value $x = 0$ for which the function $y = \frac{1}{x}$ is not defined.

Note: some integrals which involve points at which the function is not defined can be evaluated – sometimes an area which appears to be infinite does in fact have a finite value (just as the sum of an infinite geometric progression can be finite). However, it is not the case in the example above.

Since the derivative of $\ln(ax+b)$ is $\frac{a}{ax+b}$:

$$\int \frac{1}{ax+b}\,\mathrm{d}x = \frac{1}{a}\ln(ax+b)+c.$$

Example 4

Integrate

(i) $\int \frac{1}{2x+1}\,\mathrm{d}x$

(ii) $\int \frac{1}{3x}\,\mathrm{d}x$

Solution

(i) $\int \frac{1}{2x+1}\,\mathrm{d}x = \tfrac{1}{2}\ln(2x+1)+c$

(ii) $\int \frac{1}{3x}\,\mathrm{d}x = \tfrac{1}{3}\int \frac{1}{x}\,\mathrm{d}x$

$= \tfrac{1}{3}\ln|x|+c$

The example above illustrates an important point. In the same way that the integral of e^{kx} is $\frac{1}{k}\mathrm{e}^{kx}+c$, it is logical that the integral of $\frac{1}{kx}$ is $\frac{1}{k}\ln|kx|+c$, and in fact this is quite true. But in part (ii) of the example above, the integrand, $\frac{1}{3x}$ is treated as $\frac{1}{x}$ multiplied by a constant $\left(\tfrac{1}{3}\right)$, rather than as a function of a function. The result is therefore given as $\tfrac{1}{3}\ln|x|+c$ rather than as $\tfrac{1}{3}\ln|3x|+c$. The answer to this problem is that in fact these two expressions are the same. Remember that by the laws of logarithms, $\ln|3x| = \ln 3 + \ln|x|$. So $\tfrac{1}{3}\ln|3x|+c$ may be written as $\tfrac{1}{3}\ln|x|+\tfrac{1}{3}\ln 3+c$. But ln 3 is just a constant, and so it can be considered as part of the arbitrary constant.

In general, it is easier to take any constant outside the integral, as in Example 4(ii), since this gives you a simpler expression to work with. However, for a function like the one in Example 4(i) you need to use the standard result shown above.

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Integrals of trigonometric functions

The derivative of $\sin x$ is $\cos x$; the derivative of $\cos x$ is $-\sin x$. It follows that

$$\int \sin x \, dx = -\cos x + c \qquad \int \cos x \, dx = \sin x + c$$

Another useful result to bear in mind is that the derivative of $\tan x$ is $\sec^2 x$, so

$$\int \sec^2 x \, dx = \tan x + c$$

Similarly, by looking at the derivatives of $\sin kx$, $\cos kx$ and $\tan kx$, you can see that

$$\int \sin kx \, dx = -\tfrac{1}{k}\cos kx + c \qquad \int \cos kx \, dx = \tfrac{1}{k}\sin kx + c \qquad \int \sec^2 kx \, dx = \tfrac{1}{k}\tan kx + c$$

Example 5

Find (i) $\int \sin 3x \, dx$

(ii) $\int_0^{\pi/6} \sec^2 2x \, dx$

Solution

(i) $\int \sin 3x \, dx = -\tfrac{1}{3}\cos 3x + c$

(ii)
$$\int_0^{\pi/6} \sec^2 2x \, dx = \left[\tfrac{1}{2}\tan 2x\right]_0^{\pi/6}$$
$$= \tfrac{1}{2}\left(\tan \tfrac{\pi}{3} - \tan 0\right)$$
$$= \tfrac{1}{2}\sqrt{3}$$

remember that $\tan 60^\circ = \sqrt{3}$

Using trigonometric identities

Some expressions involving trigonometric functions cannot be integrated directly, but can be rewritten using a trigonometric identity, to give an expression which can be integrated.

The main ones that you should recognise are as follows:

- You can integrate $\tan^2 x$ and $\cot^2 x$ using the identities $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \operatorname{cosec}^2 x$ respectively, since both $\sec^2 x$ and $\operatorname{cosec}^2 x$ can be integrated using standard results.

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- You can integrate $\sin^2 x$ and $\cos^2 x$ using different forms of the double angle formula: $\cos 2x = 2\cos^2 x - 1$
  $= 1 - 2\sin^2 x$

- Look out for expressions like $\sin x \cos x$. You can use the double angle formula $\sin 2x = 2\sin x \cos x$ here.

- Also look out for any way in which you can simplify the expression. Sometimes, if you have an expression involving products of $\sec x$, $\operatorname{cosec} x$, $\tan x$ or $\cot x$, then writing everything in terms of $\sin x$ and $\cos x$ may result in some cancelling down being possible, which could leave you with an expression which can easily be integrated.

Remember that you may need to apply these identities to different multiples of x. For example, if you want to integrate $\cos^2 2x$, you would need to replace x by $2x$ in the formula $\cos 2x = 2\cos^2 x - 1$, giving $\cos 4x = 2\cos^2 2x - 1$ which you can then rearrange to give an expression for $\cos^2 2x$.

Using standard patterns

The chain rule allows you to differentiate a function of x by making a substitution of another variable u, say.

For example, if you want to differentiate $y = (x^2+1)^5$, you can use the substitution $u = x^2 + 1$. Then $\quad y = u^5 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = 5u^4$

and $\quad u = x^2 + 1 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$

so applying the chain rule gives $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} = 2x \times 5u^4 = 10x(x^2+1)^4$

If you are confident with the chain rule, you may perhaps write the derivative down directly in the form $5(x^2+1)^4 \times 2x$, without actually using the variable u.
Some integrations can be done by reversing this method.
The key to this is recognising functions of the form $k\mathrm{f}'(x)\mathrm{g}[\mathrm{f}(x)]$. So if you wanted to find the integral $\int x(x^2+1)^4\,\mathrm{d}x$, then you should recognise that this expression is closely related to the product of a function of x^2+1 (the expression $(x^2+1)^5$), and the derivative of x^2+1, which is $2x$. The constants will need to be adjusted, but this is straightforward.

So to deal with this integral, first adjust the constants so that you have the correct derivative:

$$\int x(x^2+1)^4\,\mathrm{d}x = \tfrac{1}{2}\int 2x(x^2+1)^4\,\mathrm{d}x$$

We now have an expression of the form $\mathrm{f}'(x)\mathrm{g}[\mathrm{f}(x)]$

Then carry out the integration.

$$\int x(x^2+1)^4\,\mathrm{d}x = \tfrac{1}{2}\times\tfrac{1}{5}(x^2+1)^5 + c$$
$$= \tfrac{1}{10}(x^2+1)^5 + c$$

You can see that this corresponds with the differentiation result above.
This method is sometimes called "integration by inspection" or "integration by recognition".

Example 6
Find $\int x^2(1+2x^3)^{\frac{1}{3}}\,\mathrm{d}x$.

Solution
The derivative of $1+2x^3$ is $6x^2$.

$$\int x^2(1+2x^3)^{\frac{1}{3}}\,\mathrm{d}x = \tfrac{1}{6}\int 6x^2(1+2x^3)^{\frac{1}{3}}\,\mathrm{d}x$$
$$= \tfrac{1}{6}\times\tfrac{3}{4}(1+2x^3)^{\frac{4}{3}} + c$$
$$= \tfrac{1}{8}(1+2x^3)^{\frac{4}{3}} + c$$

This technique can be applied to other types of function as well, not just to polynomials. The example below shows how it can be used for trigonometric functions.

Example 7
Find
$\int \sec^2 2x \tan^3 2x\ \mathrm{d}x$

Solution
The derivative of $\tan 2x$ is $2\sec^2 2x$.

$$\int \sec^2 2x\tan^3 2x\ \mathrm{d}x = \tfrac{1}{2}\int 2\sec^2 2x\tan^3 2x\ \mathrm{d}x$$
$$= \tfrac{1}{2}\times\tfrac{1}{4}\tan^4 2x + c$$
$$= \tfrac{1}{8}\tan^4 2x + c$$

One important special case of integration by recognition is when the function $\mathrm{g}(x)$ is the reciprocal function, so that the expression to be integrated is $\dfrac{k\mathrm{f}'(x)}{\mathrm{f}(x)}$.

Since the integral of $\dfrac{1}{x}$ is $\ln x$, this gives the general result

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$$

It is easy to spot integrals of this type. If the integrand is a fraction, the numerator of which is the derivative of the denominator, then the integral is the natural logarithm of the denominator. As with the earlier examples, some adjusting of constants may be needed.

Example 8

Find $\int \frac{x}{1+x^2}\,\mathrm{d}x$.

Solution

The derivative of $1 + x^2$ is $2x$.

The numerator is the derivative of the denominator.

$$\int \frac{x}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\int \frac{2x}{1+x^2}\,\mathrm{d}x$$
$$= \tfrac{1}{2}\ln\left(1+x^2\right)+c$$

Notice that modulus signs are not needed here, since $1 + x^2$ is always positive.

Be careful to distinguish between integrands of the form $\frac{k\mathrm{f}'(x)}{\mathrm{f}(x)}$ and integrands of the form $\frac{k\mathrm{f}'(x)}{\mathrm{g}[\mathrm{f}(x)]}$. The first gives a logarithm, and the second does not. The next example illustrates this.

Example 9

Find (i) $\int \frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}\,\mathrm{d}x$

(ii) $\int \frac{\mathrm{e}^{2x}}{\left(1+\mathrm{e}^{2x}\right)^3}\,\mathrm{d}x$

Solution

The derivative of $1+\mathrm{e}^{2x}$ is $2\mathrm{e}^{2x}$.

(i) $$\int \frac{\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}\,\mathrm{d}x = \tfrac{1}{2}\int \frac{2\mathrm{e}^{2x}}{1+\mathrm{e}^{2x}}\,\mathrm{d}x$$
$$= \tfrac{1}{2}\ln\left(1+\mathrm{e}^{2x}\right)+c$$

Again, since $1+\mathrm{e}^{2x}$ is always positive, no modulus signs are needed.

(ii) $$\int \frac{\mathrm{e}^{2x}}{\left(1+\mathrm{e}^{2x}\right)^3}\,\mathrm{d}x = \tfrac{1}{2}\int 2\mathrm{e}^{2x}\left(1+\mathrm{e}^{2x}\right)^{-3}\,\mathrm{d}x$$
$$= \tfrac{1}{2}\times -\tfrac{1}{2}\left(1+\mathrm{e}^{2x}\right)^{-2}+c$$
$$= -\frac{1}{4\left(1+\mathrm{e}^{2x}\right)^2}$$

The next example is a definite integration. You should always simplify logarithms as far as possible.

Example 10

Find $\int_1^2 \frac{1+x^2}{3x+x^3}\,dx$, expressing the answer as a single logarithm.

Solution

The derivative of $3x + x^3$ is $3 + 3x^2 = 3(1 + x^2)$. This is three times the numerator of the integrand.

So
$$\int_1^2 \frac{1+x^2}{3x+x^3}\,dx = \tfrac{1}{3}\int_1^2 \frac{3+3x^2}{3x+x^3}\,dx$$
$$= \tfrac{1}{3}\left[\ln\left|3x+x^3\right|\right]_1^2$$
$$= \tfrac{1}{3}(\ln 14 - \ln 4)$$
$$= \tfrac{1}{3}\ln\left(\tfrac{7}{2}\right)$$