Section 1: Solving equations numerically

Notes and Examples

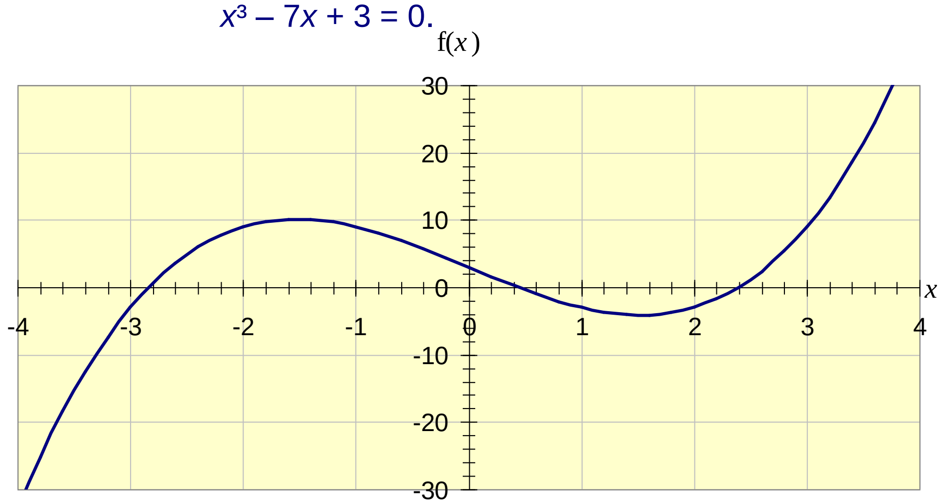
These notes contain subsections on

Change of sign methods

The equation $\mathrm{f}(x)=0$, where $\mathrm{f}(x)=x^3-7x+3$, has three real roots, but there is no simple analytical method of finding them.
The table and graph suggest first approximations to the roots of the equation

$$x^3 - 7x + 3 = 0.$$

x	f(x)
-4	-33
-3	-3
-2	9
-1	9
0	3
1	-3
2	-3
3	9
4	39



You can see from the graph and table that the roots of the equation lie in the intervals $-3<x<-2$, $0<x<1$ and $2<x<3$.

Example 1
(i) Show that the equation $x^3-7x+3=0$ has a root between x = 2 and x = 3.
(ii) Show that the root is 2.398 correct to 3 decimal places.

Solution
(i) $\mathrm{f}(x)=x^3-7x+3$
$\mathrm{f}(2)=8-14+3=-3$
$\mathrm{f}(3)=27-21+3=9$
There is a change of sign between x = 2 and x = 3, so there is a root between these two values of x.

(ii) $\mathrm{f}(2.3975) = 2.3975^3 - 7\times 2.3975 + 3 = -0.00166$

$\mathrm{f}(2.3985) = 2.3985^3 - 7\times 2.3985 + 3 = 0.00860$

There is a change of sign between x = 2.3975 and x = 2.3985, so the root lies between these two values of x, and so the root is 2.398 correct to 3 decimal places.

Change of sign methods can occasionally go wrong.
A change of sign could fail:

- where the graph of $\mathrm{f}(x)=0$ touches the x-axis, without crossing it. In this case a repeated root occurs and there is no change of sign of $\mathrm{f}(x)$ either side of the root.

  Example: $\mathrm{f}(x)=(ax-b)^2(cx+d)$: there is a root at $x=\frac{b}{a}$, but a change of sign method will not find it.

- if there is a discontinuity in the graph. In this case there may be a change of sign without a root, so the decimal search method will find a non-existent root.

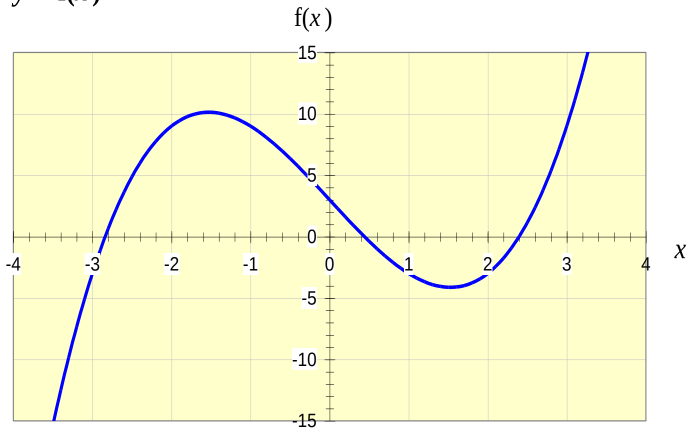
  Example: $\mathrm{f}(x)=\frac{1}{x}$: there is a change of sign, but no root, at $x=0$. The decimal search method will locate a non-existent root at $x=0$.

Fixed point iteration

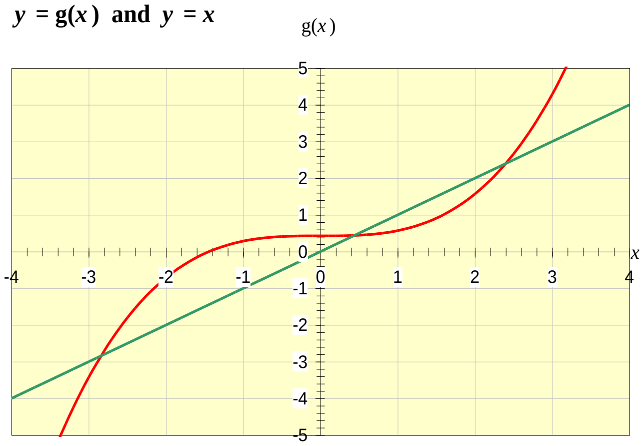
In this method, the equation $\mathrm{f}(x)=0$ is rearranged into the form $x=\mathrm{g}(x)$. Roots of the equation $x=\mathrm{g}(x)$ are therefore roots of the equation $\mathrm{f}(x)=0$.

The diagram on the left shows the graph $y=\mathrm{f}(x)$. The equation $\mathrm{f}(x)=0$ is rearranged into the form $x=\mathrm{g}(x)$, and the diagram on the right shows the functions $y=\mathrm{g}(x)$ and $y=x$. You can see that the points where the graphs cross on the right-hand diagram have the same x-coordinates as the points where the graph on the left-hand diagram cross the x-axis. These are the roots of the equation.

$y = \mathrm{f}(x)$



$y = \mathrm{g}(x)$ and $y = x$



This gives rise to the iterative formula

$$x_{n+1} = \mathrm{g}(x_n), \quad n = 0, 1, 2, 3, \ldots$$

with initial approximation x_0.

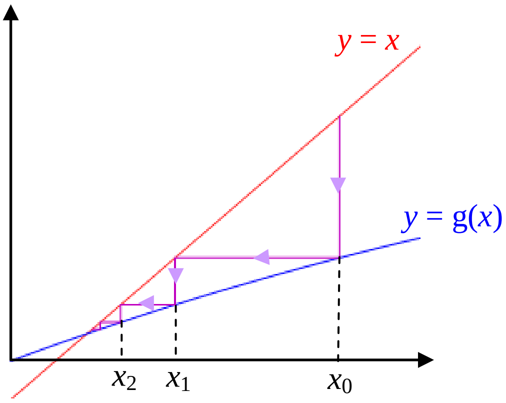
In cases where the method is successful, it works like this:

- Starting with the initial approximation x_0, you find $\mathrm{g}(x_0)$, which gives you a point on the curve $y = \mathrm{g}(x)$.
- Take this value $\mathrm{g}(x_0)$ as x_1, the new value of x, so you are moving horizontally to the line $y = x$.
- Find the value of $\mathrm{g}(x_1)$, so you are moving vertically to a new point on the curve $y = \mathrm{g}(x)$, nearer to the root
- Repeat this process until you have reached the required degree of accuracy.

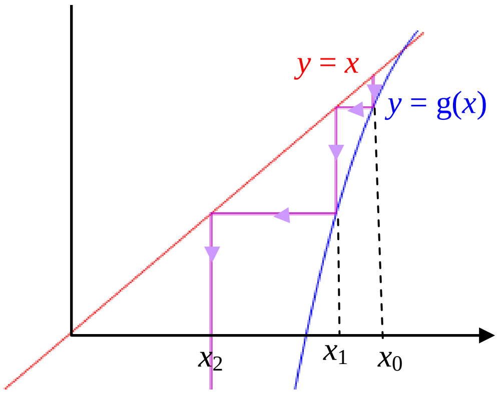
The sequence $x_0, x_1, x_2, x_3, \ldots$ will converge to a root α of the equation $x = \mathrm{g}(x)$ provided a suitable starting value x_0 is chosen and the curve is not too steep in the vicinity of the root. In fact, the curve needs to be less steep than the line $y = x$ - so the gradient needs to be between -1 and 1.

The diagrams below show some of the situations that can arise, and the significance of the steepness of the curve.

Here the gradient of the curve is positive and less than 1. The iterations converge towards the root.



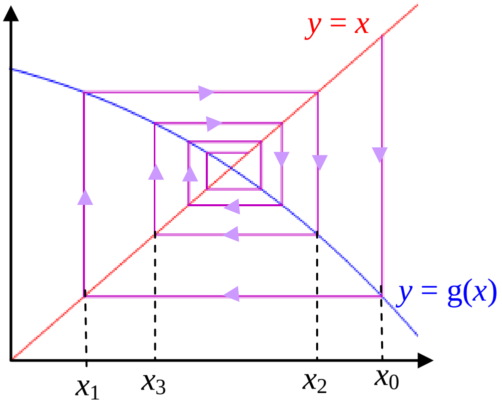
Here the gradient of the curve is positive and is greater than 1. The iterations diverge away from the root.



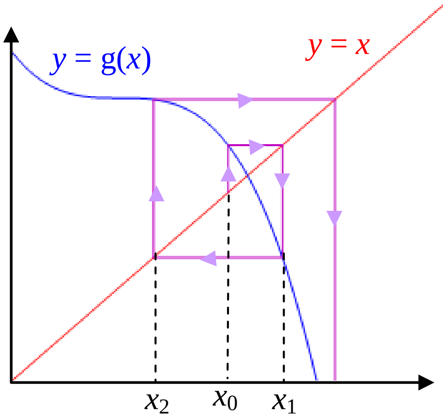
The two diagrams above are examples of staircase diagrams, in which all the iterations are on the same side of the root.

The next two diagrams are examples of cobweb diagrams, in which the iterations oscillate about the root.

Here the gradient is negative and greater than -1. The iterations oscillate about the root and converge towards it.



Here the gradient is negative and less than -1. The iterations oscillate about the root and diverge away from it.



You can usually find a root to any given degree of accuracy by continuing the iterations until the required decimal place has been unchanged for two or three iterations. However, to be absolutely certain, you need to look for a change of sign either side (see Example 2).

Example 2
Find the middle root of the equation

$$x^3 - 7x + 3 = 0$$

correct to 4 significant figures.

Solution
The graph and table at the beginning of these notes shows that the middle root lies between 0 and 1.
The equation $x^3 - 7x + 3 = 0$ may be re-arranged as follows:

$$x^3 - 7x + 3 = 0$$

$$\Rightarrow 7x = x^3 + 3$$

$$\Rightarrow x = \frac{x^3 + 3}{7}$$

The x-coordinates of the points of intersection of the graphs of $y = x$ and $y = \frac{x^3 + 3}{7}$ represent roots of the original equation $x^3 - 7x + 3 = 0$.

The rearrangement leads to the iteration

$$x_{n+1} = \frac{x_n^{\ 3} + 3}{7}, \qquad n = 0, 1, 2, 3, \ \ldots$$

To find the middle root α, let initial approximation $x_0 = 1$.

$$x_1 = \frac{x_0^{\ 3} + 3}{7} = \frac{1^3 + 3}{7} = 0.57143$$

$$x_2 = \frac{x_1^{\ 3}+3}{7} = \frac{0.57143^3+3}{7} = 0.45523$$

$$x_3 = \frac{x_2^{\ 3}+3}{7} = \frac{0.45523^3+3}{7} = 0.44205$$

$$x_4 = \frac{x_3^{\ 3}+3}{7} = \frac{0.44205^3+3}{7} = 0.44091$$

$$x_5 = \frac{x_4^{\ 3}+3}{7} = \frac{0.44091^3+3}{7} = 0.44082$$

$$x_6 = \frac{x_5^{\ 3}+3}{7} = \frac{0.44082^3+3}{7} = 0.44081$$

The sequence converges fairly quickly towards the middle root to give α = **0.4408** to 4 s.f.

Checking for a sign change:
$f(0.44075) = 0.44075^3 - 7\times0.44075 + 3 > 0$
$f(0.44085) = 0.44085^3 - 7\times0.44085 + 3 < 0$

Failure of fixed-point iteration

The success or failure of fixed-point iteration to find a particular root depends on the rearrangement being used and on the starting point. Iterations may diverge away from the root, or they may converge to a root other than the intended one. If no starting point converges to the root you want, then you need to try a different rearrangement.

Using a calculator

A calculator can be used to perform a large number of iterations quickly, using the 'Ans' key. For most calculators suitable for A level study, the process is as follows:

- Type the first approximation that you are going to use and press Enter, so that this value is used as the last answer.
- Then type in the iterative formula, using 'Ans' instead of 'x'
- Press Enter - this will give you the second approximation.
- Then press Enter again – this will give the third approximation. Keep pressing Enter to perform successive iterations for as long as you need to.

Using a spreadsheet

It is also worth experimenting with a spreadsheet to explore this topic. This has the advantage that once it is set up, you can easily change the initial value, or you can change the formulae to explore different rearrangements, If you need help in setting up a spreadsheet to do this, see the instructions below.

1. Start with a blank sheet. Use rows 1, 3, 5 and 6 for text as shown, leaving rows 2, 4 and 7 blank.

2. *Generating the first column headings*
   Highlight cell A8; type ***n***; press enter. Highlight cell B8; type ***x***; press enter.

3. *Generating the numbers in column A*
   (a) Highlight cell A9; type **0**; press enter.
   (b) Highlight cell A10; type **=A9 + 1**; press enter.
   (c) Highlight cell A10 and fill down to row 29.

4. *Generating the numbers in column B*
   (a) Highlight cell B9; type **2**; press enter.
   (b) Highlight cell B10; type **=(7*B9-3)^(1/3)**; press enter.
   (c) Highlight cell B10 and fill down to row 29; the iteration should converge towards the upper root.

5. *Generating subsequent tables*
   (a) Highlight cells A8 to B29 and copy them.
   (b) Highlight cell D8 and paste a copy of cells A8 to B29, which will occupy cells D8 to E29.
   (c) Change the first *x* value in column E to **-2**; the iteration should converge towards the lower root.
   (d) Highlight cells A8 to B29 and copy them.
   (e) Highlight cell G8 and paste a copy of cells A8 to B29, which will occupy cells G8 to H29.
   (f) Change the first *x* value in column H to **1**. Change the iteration formula in H10 to **=(H9^3+3)/7**
   (g) Highlight cell H10 and fill down to row 29; the iteration should converge towards the middle root.
   (h) Highlight cells G8 to H29 and copy them.
   (i) Highlight cell J8 and paste a copy of cells G8 to H29, which will occupy cells J8 to K29.
   (j) Change the first *x* value in column K to 3; the iteration should diverge from the upper root.

Edexcel A level Num methods 1 Notes & Examples

NUMERICAL SOLUTION OF EQUATIONS : FIXED POINT ITERATION

Finding the roots of $x^3 - 7x + 3 = 0$ by fixed point iteration

Iteration: $x_{n+1} = (7x_n - 3)^{(1/3)}$
Convergence towards upper root

n	x
0	2
1	2.223980091
2	2.324986589
3	2.367793475
4	2.385476779
5	2.392705715
6	2.395648365
7	2.396844148
8	2.397329729
9	2.397526856
10	2.397606873
11	2.397639352
12	2.397652535
13	2.397657885
14	2.397660057
15	2.397660939
16	2.397661296
17	2.397661442
18	2.397661501
19	2.397661525
20	2.397661534

Iteration: $x_{n+1} = (7x_n - 3)^{(1/3)}$
Convergence towards lower root

n	x
0	-2
1	-2.571281591
2	-2.758879119
3	-2.815229676
4	-2.831722897
5	-2.836514117
6	-2.837902919
7	-2.838305229
8	-2.838421749
9	-2.838455495
10	-2.838465268
11	-2.838468099
12	-2.838468918
13	-2.838469156
14	-2.838469224
15	-2.838469244
16	-2.83846925
17	-2.838469252
18	-2.838469252
19	-2.838469252
20	-2.838469252

Iteration: $x_{n+1} = (x_n^3 + 3)/7$
Convergence towards middle root

n	x
0	1
1	0.571428571
2	0.455226989
3	0.442048203
4	0.440911306
5	0.440816341
6	0.44080843
7	0.440807771
8	0.440807716
9	0.440807712
10	0.440807712

Iteration: $x_{n+1} = (x_n^3 + 3)/7$
Divergence from upper root

n	x
0	3
1	4.285714286
2	11.67388588
3	227.7018047
4	1686558.807
5	6.85341E+17
6	4.59855E+52
7	1.3892E+157
8	#NUM!
9	#NUM!
10	#NUM!

11	0.440807712
12	0.440807712
13	0.440807712
14	0.440807712
15	0.440807712
16	0.440807712
17	0.440807712
18	0.440807712
19	0.440807712
20	0.440807712

11	#NUM!
12	#NUM!
13	#NUM!
14	#NUM!
15	#NUM!
16	#NUM!
17	#NUM!
18	#NUM!
19	#NUM!
20	#NUM!

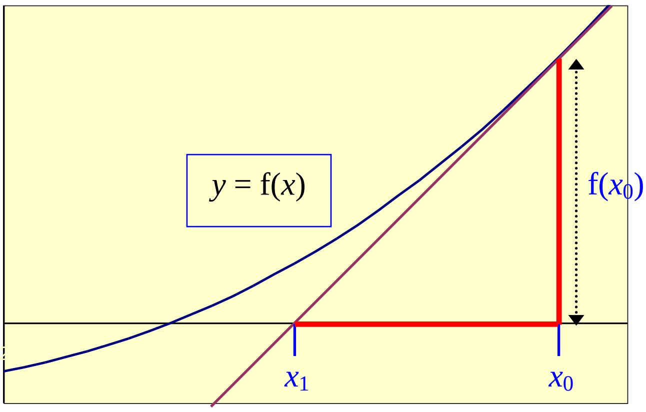
The Newton-Raphson method

This iterative formula is based on evaluating the gradient of the tangent to the curve $y = \mathrm{f}(x)$ at $x = x_0$.

From the diagram:

$$\mathrm{f}'(x_0) = \frac{\mathrm{f}(x_0)}{x_0 - x_1}$$

$$\Rightarrow \quad x_1 = x_0 - \frac{\mathrm{f}(x_0)}{\mathrm{f}'(x_0)}$$



Repeating this for x_1, x_2, etc. generates the iterative sequence:

$$x_{n+1} = x_n - \frac{\mathrm{f}(x_n)}{\mathrm{f}'(x_n)}, \quad n = 0, 1, 2, 3, \ldots$$

with initial approximation x_0.

Example 3
Use the Newton-Raphson process to find an approximation to the upper root of the equation $x^3 - 7x + 3 = 0$.

Solution
To solve the equation $x^3 - 7x + 3 = 0$, $\mathrm{f}(x) = x^3 - 7x + 3$.

$$\mathrm{f}'(x) = 3x^2 - 7$$

Hence the iteration is:

$$x_{n+1} = x_n - \frac{x_n^{\,3} - 7x_n + 3}{3x_n^{\,2} - 7}$$

To find the upper root α, let the initial approximation $x_0 = 3$:

$$x_1 = x_0 - \frac{x_0^{\ 3} - 7x_0 + 3}{3x_0^{\ 2} - 7} = 3 - \frac{3^3 - 7\times 3 + 3}{3\times 3^2 - 7} = 2.55$$

$$x_2 = x_1 - \frac{x_1^{\ 3} - 7x_1 + 3}{3x_1^{\ 2} - 7} = 2.55 - \frac{2.55^3 - 7\times 2.55 + 3}{3\times 2.55^2 - 7} = 2.411573$$

$$x_3 = x_2 - \frac{x_2^{\ 3} - 7x_2 + 3}{3x_2^{\ 2} - 7} = 2.411... - \frac{2.411..^{\ 3} - 7\times 2.411... + 3}{3\times 2.411...^{\ 2} - 7} = 2.397795$$

$$x_4 = x_3 - \frac{x_3^{\ 3} - 7x_3 + 3}{3x_3^{\ 2} - 7} = 2.397... - \frac{2.397..^{\ 3} - 7\times 2.397... + 3}{3\times 2.397...^{\ 2} - 7} = 2.397662$$

The sequence converges rapidly towards the upper root to give α = 2.398 to 4 s.f.

As for the fixed point iteration method, you can use the ‘Ans’ key to perform several iterations efficiently.

Failure of the Newton-Raphson method

The Newton-Raphson method rarely fails. However, if the starting approximation is near to a turning point, the iterations may diverge, or converge to a different root, as this means that f′(x) is small.

As for change of sign methods, the Newton-Raphson method may also fail if there is a discontinuity in the graph.

Using a spreadsheet for the Newton-Raphson method

You can use a spreadsheet for the Newton-Raphson method. The instructions are as follows:

1. Start with a blank sheet. Use rows 1, 3, 5 and 6 for text as shown, leaving rows 2, 4 and 7 blank.
2. *Generating the first column headings*
   Highlight cell A8; type ***n***; press enter. Highlight cell B8; type ***x***; press enter.
3. *Generating the numbers in column A*
   (a) Highlight cell A9; type **0**; press enter.
   (b) Highlight cell A10; type **=A9 + 1**; press enter.
   (c) Highlight cell A10 and fill down to row 19.
4. *Generating the numbers in column B*
   (a) Highlight cell B9; type **-2**; press enter.
   (b) Highlight cell B10; type **=B9 - (B9^3-7*B9+3)/(3*B9^2-7)**; press enter.
   (c) Highlight cell B10 and fill down to row 19; the iteration should converge towards the lower root.
5. *Generating subsequent tables*

(a) Highlight cells A8 to B19 and copy them.
(b) Highlight cell D8 and paste a copy of cells A8 to B19, which will occupy cells D8 to E19.
(c) Change the first *x* value in column E to **0**; the iteration should converge towards the middle root.
(d) Highlight cells A8 to B19 and copy them.
(e) Highlight cell G8 and paste a copy of cells A8 to B19, which will occupy cells G8 to H19.
(f) Change the first *x* value in column H to **3**; the iteration should converge towards the upper root.

NUMERICAL SOLUTION OF EQUATIONS: NEWTON-RAPHSON

Finding the roots of $x^3 - 7x + 3 = 0$ by the Newton-Raphson iteration

Iteration: $x_{n+1} = x_n - (x_n^3 - 7x_n + 3)/(3x_n^2 - 7)$

Convergence towards lower root

n	*x*
0	-2
1	-3.8
2	-3.10418502
3	-2.86762545
4	-2.83888173
5	-2.83846934
6	-2.83846925
7	-2.83846925
8	-2.83846925
9	-2.83846925
10	-2.83846925

Convergence towards middle root

n	*x*
0	1
1	0.25
2	0.435779817
3	0.440802552
4	0.440807711
5	0.440807712
6	0.440807712
7	0.440807712
8	0.440807712
9	0.440807712
10	0.440807712

Convergence towards upper root

n	*x*
0	3
1	2.55
2	2.411573056
3	2.397795305
4	2.397661553
5	2.397661541
6	2.397661541
7	2.397661541
8	2.397661541
9	2.397661541
10	2.397661541

Finding the single root of $5/x^3 + 3 = 0$ by the Newton-Raphson iteration
Solution represents the cube root of (-5/3)

Iteration: $x_{n+1} = x_n - (5/x_n^3 + 3)/(-15/x_n^4)$

Convergence towards root

n	*x*
0	-1
1	-1.13333333
2	-1.1811516
3	-1.18559734
4	-1.1856311
5	-1.1856311
6	-1.1856311

Divergence from root

n	*x*
0	-2
1	0.533333333
2	0.72729284
3	1.025682438
4	1.588927678
5	3.393383334
6	31.0437885

7	-1.1856311
8	-1.1856311
9	-1.1856311
10	-1.1856311

7	185791.4075
8	2.38305E+20
9	6.45E+80
10	#NUM!