Section 1: Introduction to differential equations

Section test

1. The deceleration of a car travelling velocity v m s^{-1} at time t s is proportional to the square of the velocity. If k is a positive constant, this situation can be modelled by the differential equation

(a) $-\frac{\mathrm{d}v}{\mathrm{d}t} = k + v^2$ (b) $\frac{\mathrm{d}v}{\mathrm{d}t} = -kv^2$

(c) $\frac{\mathrm{d}v}{\mathrm{d}t} = kv^2$ (d) $\frac{\mathrm{d}t}{\mathrm{d}v} = -kv^2$

2. The mass m of a substance at time t minutes is decreasing at a rate which is inversely proportional to the cube root of the mass. If k is a positive constant, then

(a) $\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{k}{\sqrt[3]{m}}$ (b) $\frac{\mathrm{d}m}{\mathrm{d}t} = -\frac{k}{m^3}$

(c) $\frac{\mathrm{d}m}{\mathrm{d}t} = \frac{k}{\sqrt[3]{m}}$ (d) $\frac{\mathrm{d}m}{\mathrm{d}t} = -k\sqrt[3]{m}$

3. Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x$, and when $x = 0$, $y = 0$, then

(a) $y = 1 - \cos x$ (b) $y = -\cos x$

(c) $y = 1 + \cos x$ (d) $y = \cos x$

4. The general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y^3}$ is

(a) $y = \sqrt[4]{2x^2}$ (b) $y = \sqrt[4]{2x^2} + k$

(c) $y = \sqrt[4]{2x^2 + k}$ (d) $y = 2x^2 + k$

5. The solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{xy}$ which passes through the point (4, 1) is

(a) $y = \frac{1}{9}\left(x^3 - 55\right)$ (b) $y = \left(\frac{3}{2}x^{\frac{3}{2}} - 11\right)^2$

(c) $y = \left(\frac{1}{3}x^{\frac{3}{2}} - 1\right)^2$ (d) $y = \frac{1}{9}(x^{\frac{3}{2}} - 5)^2$

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6. The general solution of the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = x\mathrm{e}^{-2y}$ is

(a) $y = \frac{1}{2}\ln\left(\frac{1}{4}x^2\right) + c$

(b) $y = \ln|x| + c$

(c) $y = \frac{1}{2}\ln\left(\frac{1}{4}x^2 + c\right)$

(d) $y = \frac{1}{2}\ln(x^2 + c)$

7. Given that at time t, $\frac{\mathrm{d}x}{\mathrm{d}t} = x^2 t$ and the initial value of x is 1, then

(a) $x = \frac{2}{2 - t^2}$

(b) $x = 1 - \frac{2}{t^2}$

(c) $x = 1 - t^2$

(d) $x = \frac{2}{2 + t^2}$

8. The volume V cm^3 of a balloon at time t minutes is decreasing at a rate proportional to the cube root of V. The constant of proportionality is 1. Given that the initial volume is 8 cm^3, then

(a) $V = \left(4 + \frac{2}{3}t\right)^{\frac{3}{2}}$

(b) $V = \left(4 - \frac{2}{3}t\right)^{\frac{3}{2}}$

(c) $V = 8 - \left(\frac{2}{3}t\right)^{\frac{3}{2}}$

(d) $V = 8 + \left(\frac{2}{3}t\right)^{\frac{3}{2}}$

9. Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{xy^2}{x+1}$, for $x \geq 0$, and when $x = 0$, $y = 1$, then

(a) $y = \frac{1}{1 + x - \ln(1+x)}$

(b) $y = \frac{1}{x - \ln(1+x)}$

(c) $y^2 = \mathrm{e}^{-x}(x+1)$

(d) $y^3 = 3x - 3\ln(1+x) + 1$

10. A population of mice increases such that its rate of increase is proportional to the size of the population.
Initially there are 20 mice, and the population is increasing at the rate of 4 mice per month.
After t months the size of the population, P, is given by

(a) $P = 20\mathrm{e}^{5t}$

(b) $P = \mathrm{e}^{5t} + 19$

(c) $P = 20\mathrm{e}^{t/5}$

(d) $P = \mathrm{e}^{t/5} + 19$

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Solutions to section test

1. Deceleration is rate of change of velocity $= \frac{dv}{dt}$.

   Deceleration is proportional to square of velocity, so $\frac{dv}{dt} \propto v^2$

   Since deceleration is negative, the constant of proportionality must be negative.

   The differential equation is therefore $\frac{dv}{dt} = -kv^2$, where k is a positive constant.

2. Rate of change of mass is given by $\frac{dm}{dt}$.

   $\frac{dm}{dt} \propto \frac{1}{\sqrt[3]{m}}$

   Since the mass is decreasing, the constant of proportionality is negative.

   Therefore $\frac{dm}{dt} = -\frac{k}{\sqrt[3]{m}}$

3. $\frac{dy}{dx} = \sin x$

   $y = \int \sin x \, dx = -\cos x + c$

   When $x = 0$, $y = 0 \Rightarrow 0 = -\cos 0 + c \Rightarrow 0 = -1 + c \Rightarrow c = 1$

   $y = 1 - \cos x$

4. $\frac{dy}{dx} = \frac{x}{y^3}$

   $\int y^3 \, dy = \int x \, dx$

   $\frac{1}{4} y^4 = \frac{1}{2} x^2 + c$

   $y^4 = 2x^2 + 4c$

   $y = \sqrt[4]{2x^2 + k}$

   Replacing $4c$ by k

5. $\frac{dy}{dx} = \sqrt{xy} = x^{\frac{1}{2}}y^{\frac{1}{2}}$

$\int y^{-\frac{1}{2}}\,dy = \int x^{\frac{1}{2}}\,dx$

$2y^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}} + c$

When $x = 4,\ y = 1 \Rightarrow 2\sqrt{1} = \frac{2}{3}\times 4^{\frac{3}{2}} + c$

$\Rightarrow 2 = \frac{16}{3} + c$

$\Rightarrow c = -\frac{10}{3}$

$2y^{\frac{1}{2}} = \frac{2}{3}x^{\frac{3}{2}} - \frac{10}{3}$

$\sqrt{y} = \frac{1}{3}\left(x^{\frac{3}{2}} - 5\right)$

$y = \frac{1}{9}\left(x^{\frac{3}{2}} - 5\right)^2$

6. $\frac{dy}{dx} = xe^{-2y}$

$\int e^{2y}\,dy = \int x\,dx$

$\frac{1}{2}e^{2y} = \frac{1}{2}x^2 + k$

$e^{2y} = x^2 + 2k$

$2y = \ln\left(x^2 + c\right)$

$y = \frac{1}{2}\ln\left(x^2 + c\right)$

7. $\frac{dx}{dt} = x^2 t$

$\int x^{-2}\,dx = \int t\,dt$

$-x^{-1} = \frac{1}{2}t^2 + c$

When $t = 0,\ x = 1 \Rightarrow -1 = 0 + c \Rightarrow c = -1$

$-x^{-1} = \frac{1}{2}t^2 - 1$

$\frac{1}{x} = 1 - \frac{1}{2}t^2 = \frac{2 - t^2}{2}$

$x = \frac{2}{2 - t^2}$

8. $\frac{dV}{dt} = -\sqrt[3]{V} = -V^{\frac{1}{3}}$

$\int V^{-\frac{1}{3}}\,dV = \int -1\,dt$

$\frac{3}{2}V^{\frac{2}{3}} = -t + c$

When $t = 0,\ V = 8 \Rightarrow \frac{3}{2}\times 8^{\frac{2}{3}} = c \Rightarrow c = \frac{3}{2}\times 4 = 6$

$\frac{3}{2}V^{\frac{2}{3}} = -t + 6$

$V^{\frac{2}{3}} = 4 - \frac{2}{3}t$

$V = \left(4 - \frac{2}{3}t\right)^{\frac{3}{2}}$

9. $\frac{dy}{dx} = -\frac{xy^2}{x+1}$

$\int -y^{-2}dy = \int \frac{x}{x+1}dx$

$y^{-1} = \int\left(\frac{x+1}{x+1} - \frac{1}{x+1}\right)dx$

$= \int\left(1 - \frac{1}{x+1}\right)dx$

$= x - \ln(1+x) + c$

$y = \frac{1}{x - \ln(1+x) + c}$

When $x = 0$, $y = 1 \Rightarrow 1 = \frac{1}{0 - \ln 1 + c} \Rightarrow c = 1$

$y = \frac{1}{x - \ln(1+x) + 1}$

10. $\frac{dP}{dt} = kP$

Initially $P = 20$ and $\frac{dP}{dt} = 4 \quad \Rightarrow 4 = 20k \quad \Rightarrow k = \frac{1}{5}$

$\frac{dP}{dt} = \frac{1}{5}P$

$\int \frac{1}{P}dP = \int \frac{1}{5}\,dt$

$\ln P = \frac{1}{5}t + c$

When $t = 0$, $P = 20 \Rightarrow \ln 20 = c$

$\ln P = \frac{1}{5}t + \ln 20$

$\ln \frac{P}{20} = \frac{1}{5}t$

$\frac{P}{20} = e^{\frac{1}{5}t}$

$P = 20e^{\frac{1}{5}t}$