Section 1: The shape of curves

Notes and Examples

These notes contain sub-sections on:
- **Concave and convex**
- **Stationary points of inflection**
- **Non-stationary points of inflection**

Concave and convex

You already know that the first derivative of a curve tells you about the gradient of the curve. The gradient of a curve tells you something about its shape: where the gradient is positive, the function is increasing, and where it is negative, the function is decreasing.

The second derivative gives you further information about the shape of a curve. The second derivative tells you about the rate of change of the gradient.

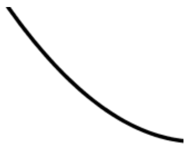
If the second derivative is positive, the gradient is increasing (this may mean it is getting ‘more positive’ or ‘less negative’).

Gradient positive and getting more positive

$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$

Gradient negative and getting less negative

$\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$

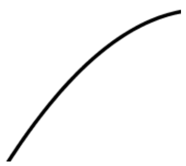


The two curve sections shown above, for which $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0$, are described as **convex** (or sometimes as **concave upwards**). Notice that if you draw a chord connecting two points on a convex section of curve, the chord is always above the curve. The tangent to the curve always lies below the curve.

If the second derivative is negative, the gradient is decreasing (this may mean it is getting ‘less positive’ or ‘more negative’).

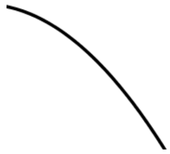
Gradient positive and getting less positive

$\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ and $\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0$



Gradient negative and getting more negative

$\frac{dy}{dx} < 0$ and $\frac{d^2y}{dx^2} < 0$



The two curve sections shown above, for which $\frac{d^2y}{dx^2} < 0$, are described as **concave** (or sometimes as **concave downwards**). Notice that if you draw a chord connecting two points on a concave section of curve, the chord is always below the curve. The tangent to the curve always lies above the curve.

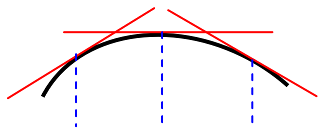
Stationary points of inflection

You have already met the idea that points on a curve where the tangent is horizontal are called *stationary points.*

At these points, the gradient of the curve is zero, so $\frac{dy}{dx} = 0$.

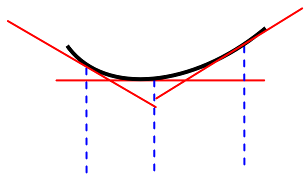
You have already come across two different types of stationary point:

Local **maximum**
The gradient is positive to the left, zero at the point, and negative to the right.
If $\frac{d^2y}{dx^2}$ < 0, the gradient function $\frac{dy}{dx}$ is decreasing.
At a maximum point, the gradient goes from + to 0 to –, in other words is decreasing.
The curve is concave.



So $\frac{d^2y}{dx^2} < 0 \Rightarrow$ the stationary point is a maximum.

Local **minimum**
The gradient is negative to the left, zero at the point, and positive to the right.
If $\frac{d^2y}{dx^2}$ > 0, the gradient function $\frac{dy}{dx}$ is increasing.
At a minimum point, the gradient goes from – to 0 to +, in other words is increasing.
The curve is convex.



So $\frac{d^2y}{dx^2} > 0 \Rightarrow$ the stationary point is a minimum.

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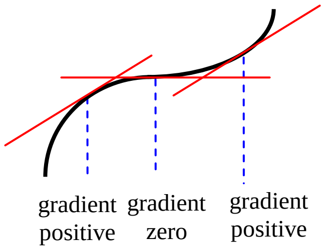
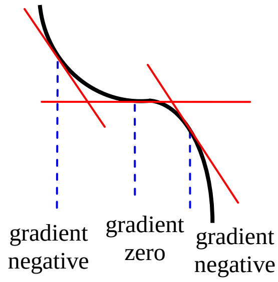
It may already have occurred to you to wonder what is happening if the gradient has the same sign on both sides of the stationary point.
This could mean that:

- The gradient goes from positive to zero to positive.
  The curve is concave before the stationary point, and convex after it.
  The second derivative goes from negative to positive.



Or:

- The gradient goes from negative to zero to negative.
  The curve is convex before the stationary point, and concave after it.
  The second derivative goes from positive to negative.



A stationary point like this, for which the gradient has the same sign either side of the stationary point, is called a **stationary point of inflection**.

Notice that at the stationary point, the tangent to the curve passes through the curve, so that it is above the curve on one side of the stationary point, and below it on the other.

At a stationary point of inflection, the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is zero.
However, the converse is not true: if the second derivative is zero, the stationary point could be a maximum point or a minimum point. (To convince yourself of this, investigate the curves $y = x^4$, $y = -x^4$ and $y = x^5$).

So if you try to use the second derivative test to find out the nature of a stationary point, and the second derivative turns out to be zero, you need to investigate further using one of the following:

- If the sign of the gradient (the first derivative) is the same on both sides of the stationary point, then it is a stationary point of inflection
- If the sign of the second derivative changes, then it is a stationary point of inflection.

Example 1
Find and classify the stationary points on the curve $y = x^4 - x^3 - 3x^2 + 5x - 2$.
Sketch the curve.

Solution

$$y = x^4 - x^3 - 3x^2 + 5x - 2$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 4x^3 - 3x^2 - 6x + 5$$

At stationary points $4x^3 - 3x^2 - 6x + 5 = 0$
By inspection $x = 1$ is a root, so $(x-1)$ is a factor

$$4x^3 - 3x^2 - 6x + 5 = 0$$

$$(x-1)(4x^2 + x - 5) = 0$$

$$(x-1)(x-1)(4x+5) = 0$$

$$x = 1 \text{ or } -\tfrac{5}{4}$$

When $x = 1, y = 0$

When $x = -\tfrac{5}{4}, y = -\tfrac{2187}{256}$

$$\frac{d^2y}{dx^2} = 12x^2 - 6x - 6$$

When $x = 1$, $\dfrac{d^2y}{dx^2} = 0$

Since the second derivative is zero at this stationary point, you cannot use the second derivative test to determine the nature of the stationary point. Instead, test the gradient either side

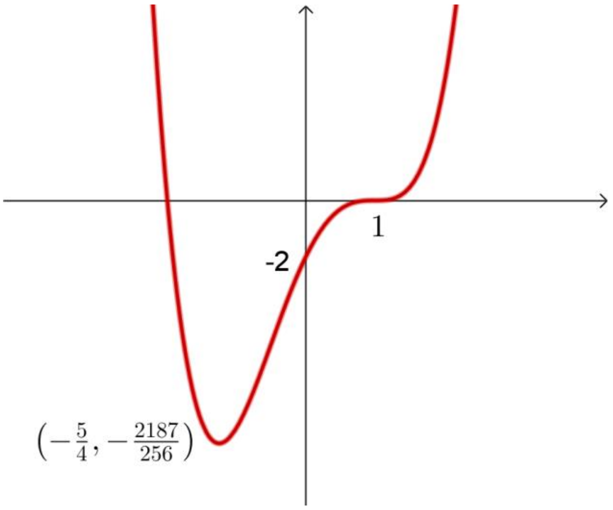
When $x < 1$, $\dfrac{dy}{dx} > 0$

When $x > 1$, $\dfrac{dy}{dx} > 0$

So (1, 0) is a stationary point of inflection

When $x = -\tfrac{5}{4}$, $\dfrac{dy}{dx} > 0$

So $\left(-\tfrac{5}{4}, -\tfrac{2187}{256}\right)$ is a minimum point.



In the example above, the original curve $y = x^4 - x^3 - 3x^2 + 5x - 2$ can be written in factorised form as $y = (x-1)^3(x+2)$. This gives a clue about the stationary point – if the root $x = 1$ occurred twice, then it would just touch the x-axis, but as it occurs a third time, the curve turns again, giving a point of inflection.

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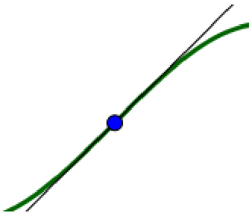
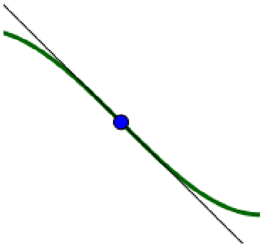
Non-stationary points of inflection

You have seen above that a stationary point of inflection is a point where $\frac{dy}{dx}=0$ and $\frac{d^2y}{dx^2}=0$, and the sign of $\frac{d^2y}{dx^2}$ changes as the curve passes through the stationary point.

It is possible to have points of inflection that are not stationary points: they are points for which $\frac{d^2y}{dx^2}=0$, and the sign of $\frac{d^2y}{dx^2}$ changes as the curve passes through the point.

Here the curve changes from convex to concave.
The second derivative goes from positive to negative: initially the gradient is becoming more positive and then it is becoming less positive.





Here the curve changes from concave to convex.
The second derivative goes from negative to positive: initially the gradient is becoming more negative and then it is becoming less negative.

As with stationary points of inflection, you need to be careful when identifying them. At a point of inflection (whether stationary or non-stationary), $\frac{d^2y}{dx^2}=0$, but the converse is not true. You should check that the second derivative changes sign. For example, investigate the curve $y=x^4+x+1$ at the point where $x=0$. You will find that $\frac{d^2y}{dx^2}=0$ when $x=0$, but $\frac{d^2y}{dx^2}>0$ on both sides of the point, so it is not a point of inflection.

Example 2
Find the points of inflection on the curve $y=3x^5-5x^4+2$

Solution

$$y=3x^5-5x^4+2$$

$$\frac{dy}{dx}=15x^4-20x^3$$

$$\frac{d^2y}{dx^2}=60x^3-60x^2=60x^2(x-1)$$

$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ when $x = 0$ and when $x = 1$

On both sides of $x = 0$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$ so this is not a point of inflection

At $x = 1$, $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ changes from negative to positive

So (1, 0) is the only point of inflection.