Section 3: Implicit differentiation

Notes and Examples

These notes contain subsections on

Differentiating functions of *y* with respect to *x*

Suppose that y is a function of x, say x^2. You can differentiate y to get $2x$; but what about differentiating y^2, or $\sin y$ or $\ln y$? You could convert them to functions of x, i.e. x^4, $\sin x^2$ or $\ln x^2$; but can you find the derivative of these functions without converting into functions of x?

For example, suppose $y = x^2$ as above and you want to find the derivative of y^2 with respect to x. Let $u = y^2$. You want to find $\frac{du}{dx}$

$$u = y^2 \Rightarrow \frac{du}{dy} = 2y$$

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x.$$

Using the chain rule:

$$\begin{aligned}\frac{du}{dx} &= \frac{du}{dy} \times \frac{dy}{dx} \\ &= 2y\frac{dy}{dx} \\ &= 2x^2 \times 2x \\ &= 4x^3 \qquad \text{(which is indeed the derivative of } x^4\text{)}\end{aligned}$$

The other functions of y can be differentiated in the same way:

To differentiate $\sin y$ with respect to x, let $u = \sin y$. You want to find $\frac{du}{dx}$.

$$u = \sin y \Rightarrow \frac{du}{dy} = \cos y$$

$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

Using the chain rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$
$$= \cos y \frac{dy}{dx}$$
$$= 2x \cos y$$

To differentiate $\ln y$ with respect to x, let $u = \ln y$. You want to find $\frac{du}{dx}$.

$$u = \ln y \Rightarrow \frac{du}{dy} = \frac{1}{y}$$
$$y = x^2 \Rightarrow \frac{dy}{dx} = 2x$$

Using the chain rule:

$$\frac{du}{dx} = \frac{du}{dy} \times \frac{dy}{dx}$$
$$= \frac{1}{y} \frac{dy}{dx}$$
$$= \frac{1}{x^2} \times 2x$$
$$= \frac{2}{x}$$

This idea can be generalised:

If y is a function of x, then $\frac{d}{dx}[f(y)] = \frac{df}{dy} \times \frac{dy}{dx}$.

Differentiating implicit functions

An **implicit function** is an equation between two variables, say x and y, in which y is not given explicitly as a function of x. The equation of a circle is an example of an implicit function.

For example, $x^2 + y^2 = 25$ is the equation of a circle, centre O and radius 5.

You can differentiate an implicit function like this by differentiating each side of the equation with respect to x:

$$x^2 + y^2 = 25$$

$\Rightarrow$ $$\frac{d}{dx}[x^2] + \frac{d}{dx}[y^2] = \frac{d}{dx}[25]$$

$\Rightarrow$ $$2x + 2y\frac{dy}{dx} = 0$$

$\frac{d}{dx}[y^2] = 2y\frac{dy}{dx}$, as before.

You can now find $\frac{dy}{dx}$ in terms of x and y by rearranging this equation:

$$2x + 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \quad 2y\frac{dy}{dx} = -2x$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

This result enables you, for example, to find the gradient at the point (3, 4) of the circle. Here, $x = 3$ and $y = 4$, so $\frac{dy}{dx} = -\frac{x}{y} = -\frac{3}{4}$.

Example 1

Show that the curve $y + y^3 = 2\sin x$ has a turning point at the point $\left(\frac{\pi}{2}, 1\right)$.

Solution

$y + y^3 = 2\sin x$

Differentiating implicitly gives:

$$\frac{d}{dx}[y + y^3] = \frac{d}{dx}[2\sin x]$$

$$\Rightarrow \quad \frac{dy}{dx} + 3y^2\frac{dy}{dx} = 2\cos x\,,$$

$$\Rightarrow \quad \frac{dy}{dx}(1 + 3y^2) = 2\cos x$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\cos x}{1 + 3y^2}$$

At the point $\left(\frac{\pi}{2}, 1\right)$ $x = \frac{\pi}{2}$ and $y = 1$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{2\cos\frac{\pi}{2}}{1 + 3} = 0$$

So $\left(\frac{\pi}{2}, 1\right)$ is a turning point.

Differentiating products of *x* and *y*

What is the derivative of xy^2 with respect to x?

You can use the product rule to do this.

$u = x \Rightarrow \frac{du}{dx} = 1$ and $v = y^2 \Rightarrow \frac{dv}{dx} = 2y\frac{dy}{dx}$.

So $\frac{d}{dx}(xy^2) = u\frac{dv}{dx} + v\frac{du}{dx}$

$$= 2xy\frac{dy}{dx} + y^2$$

Example 2

Given that $y^2 + 3xy + x^3 = 25$, find $\frac{dy}{dx}$ in terms of x and y.

Solution

$y^2 + 3xy + x^3 = 25$

Differentiating implicitly gives:

$$\frac{d}{dx}[y^2 + 3xy + x^3] = \frac{d}{dx}[25]$$

$$\Rightarrow \quad 2y\frac{dy}{dx} + 3x\frac{dy}{dx} + 3y + 3x^2 = 0$$

$$\Rightarrow \quad (2y + 3x)\frac{dy}{dx} = -3y - 3x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = -\frac{(3y + 3x^2)}{2y + 3x}$$

The product rule is used here to differentiate $3xy$, with $u = 3x$ and $v = y$. It is not necessary to write out all the details of the product rule if you don't feel that you need to.

Logarithmic differentiation

The technique of implicit differentiation can be used to simplify differentiating complicated product and quotient functions, as in the following example.

Example 3

Find the derivative of $y = \frac{(1+x)(1-2x)^3}{\sqrt{1+x^2}}$

Solution

$$y = \frac{(1+x)(1-2x)^3}{\sqrt{1+x^2}}$$

Take logarithms of each side:

$$\ln y = \ln\left[\frac{(1+x)(1-2x)^3}{\sqrt{1+x^2}}\right]$$

$$= \ln(1+x) + \ln(1-2x)^3 - \ln\sqrt{1+x^2}$$

$$= \ln(1+x) + 3\ln(1-2x) - \frac{1}{2}\ln(1+x^2)$$

Now differentiate implicitly:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{x+1} + \frac{3}{1-2x} \times (-2) - \frac{1}{2}\frac{2x}{(1+x^2)}$$

$$\Rightarrow \quad \frac{dy}{dx} = y\left(\frac{1}{x+1} - \frac{6}{1-2x} - \frac{x}{(1+x^2)}\right)$$

$$= \frac{(1+x)(1-2x)^3}{\sqrt{1+x^2}}\left(\frac{1}{x+1} - \frac{6}{1-2x} - \frac{x}{(1+x^2)}\right)$$

This is quite a fearsome looking expression! But try differentiating *y* directly and you will appreciate that the logarithmic differentiation is considerably easier!