Section 2: Parametric differentiation and integration

Notes and Examples

These notes contain subsections on

Finding the gradient of a curve given by parametric equations

You can use the **chain rule**

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

to find the **gradient function**, $\frac{dy}{dx}$, of a curve defined by parametric equations.

The chain rule can be rewritten as

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \quad \frac{dx}{dt} \neq 0$$

Multiplying by $\frac{dt}{dx}$ is the same as dividing by $\frac{dx}{dt}$

Dividing by 0 is undefined.

Example 1

A circle has the parametric equations
$$x = 2 + 5\cos\theta$$
$$y = 4 + 5\sin\theta$$

Find the gradient of the circle at the point with parameter θ.

Solution

$$x = 2 + 5\cos\theta \Rightarrow \frac{dx}{d\theta} = -5\sin\theta$$

$$y = 4 + 5\sin\theta \Rightarrow \frac{dy}{d\theta} = 5\cos\theta$$

Using the chain rule $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$

$$\frac{dy}{dx} = \frac{\cancel{5}\cos\theta}{-\cancel{5}\sin\theta}$$

So $\frac{dy}{dx} = \frac{-\cos\theta}{\sin\theta} = -\cot\theta$

Finding the equation of the tangent and normal to a curve

Finding $\frac{\mathrm{d}y}{\mathrm{d}x}$ gives you the gradient of the tangent to the curve at the point with parameter t (or θ).
You can then use $y - y_1 = m(x - x_1)$ to find the **equation of the tangent.**

The **gradient of the normal** is $\dfrac{-1}{\mathrm{d}y/\mathrm{d}x}$ since the tangent and normal are perpendicular to each other.

$$m_{\text{normal}} = \frac{-1}{m_{\text{tangent}}}$$

Example 2

An ellipse is defined by the parametric equations $\begin{array}{l} x = 4\cos\theta \\ y = 2\sin\theta \end{array}$

(a) Find the equation of the tangent to the ellipse at the point with parameter θ.

(b) Find the equation of the normal to the ellipse at the point $(2, \sqrt{3})$

Solution

(a) First you need to find the gradient function of the curve

$$x = 4\cos\theta \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}\theta} = -4\sin\theta$$

$$y = 2\sin\theta \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}\theta} = 2\cos\theta$$

Using the chain rule $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\mathrm{d}y/\mathrm{d}\theta}{\mathrm{d}x/\mathrm{d}\theta}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2\cos\theta}{-4\sin\theta}$$

So $\quad \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{-\cos\theta}{2\sin\theta}$

Now use $y - y_1 = m(x - x_1)$ where $x_1 = 4\cos\theta$, $y_1 = 2\sin\theta$ and $m = \dfrac{-\cos\theta}{2\sin\theta}$ to find the equation of the tangent.

So: $\quad y - 2\sin\theta = \dfrac{-\cos\theta}{2\sin\theta}(x - 4\cos\theta)$

Multiply both sides by $2\sin\theta$: $\quad 2y\sin\theta - 4\sin^2\theta = -\cos\theta(x - 4\cos\theta)$

Expanding the brackets: $\quad 2y\sin\theta - 4\sin^2\theta = -x\cos\theta + 4\cos^2\theta$

Rearranging: $\quad 2y\sin\theta + x\cos\theta = 4\cos^2\theta + 4\sin^2\theta$

Now $\cos^2\theta + \sin^2\theta \equiv 1$ so $4\cos^2\theta + 4\sin^2\theta \equiv 4$

So the equation of the tangent is $2y\sin\theta + x\cos\theta = 4$

(b) We need to find the value of the parameter at the point $(2, \sqrt{3})$

Now the curve is $\begin{aligned} x &= 4\cos\theta \\ y &= 2\sin\theta \end{aligned}$

So solving $\quad 4\cos\theta = 2 \Rightarrow \cos\theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{5\pi}{3}$

and $\quad 2\sin\theta = \sqrt{3} \Rightarrow \sin\theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \frac{\pi}{3}, \frac{2\pi}{3}$

so the value of the parameter at $(2, \sqrt{3})$ is $\theta = \frac{\pi}{3}$

$$m_{\text{normal}} = \frac{-1}{m_{\text{tangent}}}$$

The gradient of the tangent is $\frac{dy}{dx} = \frac{-\cos\theta}{2\sin\theta}$ from part (a)

So the gradient of the normal is $\frac{2\sin\theta}{\cos\theta}$

When $\theta = \frac{\pi}{3}$ the gradient of the normal is $\frac{2\sin\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{3}\right)} = \frac{2\left(\frac{\sqrt{3}}{2}\right)}{\frac{1}{2}} = 2\sqrt{3}$

Now use $y - y_1 = m(x - x_1)$ where $x_1 = 2$, $y_1 = \sqrt{3}$ and $m = 2\sqrt{3}$ to find the equation of the tangent.

So $\quad y - \sqrt{3} = 2\sqrt{3}(x - 2)$

Expanding the brackets $\quad y - \sqrt{3} = 2\sqrt{3}x - 4\sqrt{3}$

Simplifying $\quad y = 2\sqrt{3}x - 3\sqrt{3}$

In the example above, notice that in part (i) the general equation of the tangent was found, in terms of the parameter θ. In part (ii), the equation of the tangent at a specific point was found.

Finding the turning points of a curve

At a turning point $\frac{dy}{dx} = 0$.

So to find the turning points

- Find an expression for the gradient function $\frac{dy}{dx}$
- Put your expression equal to 0
- Solve the equation to find the value of the parameter at the turning point

You can identify the nature of any turning points by examining the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ just before and after the turning point.

Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$		
+ve	0	-ve
/	—	\
maximum		

Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$		
-ve	0	+ve
\	—	/
minimum		

Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$		
+ve	0	+ve
/	—	/
-ve	0	-ve
\	—	\
Point of inflection		

Example 3

Find the turning points of the curve defined by the parametric equations $x = t - 1, \quad y = t^4 - 2t^3$ and identify their nature.

Solution

$$x = t - 1 \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 1$$

$$y = t^4 - 2t^3 \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}t} = 4t^3 - 6t^2$$

Using the chain rule $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}x/\mathrm{d}t} = 4t^3 - 6t^2$

At a turning point $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

So $\quad 2t^2(2t - 3) = 0$

$\Rightarrow \quad t = 0$ or $t = \frac{3}{2}$

So there are turning points at the points with parameters $t = 0$ or $t = \frac{3}{2}$

Substitute $t = 0$ and $t = \frac{3}{2}$ into the parametric equations $x = t - 1, \quad y = t^4 - 2t^3$ to find the coordinates of the turning points:

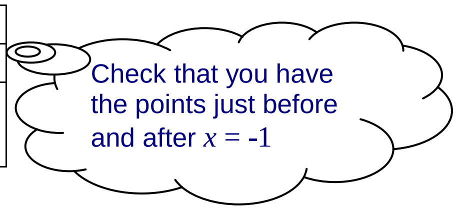
At $t = 0$: $\quad x = -1$ and $y = 0 \Rightarrow (-1, 0)$ is a turning point

At $t = \frac{3}{2}$: $\quad x = \frac{1}{2}$ and $y = -\frac{27}{16} \Rightarrow \left(\frac{1}{2}, -\frac{27}{16}\right)$ is a turning point

Now examine the sign of $\frac{\mathrm{d}y}{\mathrm{d}x} = 4t^3 - 6t^2$ just before and after each turning point.

At $t = 0 \Rightarrow x = -1$

Value of t	$t = -0.1$	$t = 0$	$t = 0.1$
Value of x	-1.1	$x = -1$	-0.9
Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$	-ve \	0 —	-ve \



At $t = \frac{3}{2} \Rightarrow x = \frac{1}{2}$

Value of t	$t = 1.4$	$t = 1.5$	$t = 1.6$
Value of x	0.4	$x = 0.5$	0.6
Sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$	-ve \	0	+ve /

So there is a point of inflection at (-1, 0) and a minimum at $\left(\frac{1}{2}, -\frac{27}{16}\right)$

For some interesting extension work, try finding and identifying the stationary points of the graph $\begin{matrix} x = t^2 - 1 \\ y = t^4 - 2t^3 \end{matrix}$. Then use graphing software to sketch the graph. You may be surprised by the result!

Finding areas

In AS Maths you learnt to find areas using integration.

The area under a curve from $x = a$ to $x = b$ is given by $\int_a^b y \, \mathrm{d}x$.

When working with parametric equations, you can use the chain rule so that the variable involved is the parameter:

$$\text{Area } \int y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t$$

It is important to remember that the limits of integration must be values of t, not x.

Example 4

(i) Find the values of t at which the curve $x = 3t + 2,\ y = 1 - t^2$ meets the x-axis.

(ii) Find the area enclosed between the curve and the x-axis.

Solution

(i) When the curve meets the x-axis, $y = 0 \Rightarrow 1 - t^2 = 0 \quad \Rightarrow t = \pm 1$

(ii) $x = 3t + 2 \quad \Rightarrow \frac{\mathrm{d}x}{\mathrm{d}t} = 3$

$$\begin{aligned} \text{Area } &= \int_{-1}^{1} y \frac{\mathrm{d}x}{\mathrm{d}t} \mathrm{d}t \\ &= \int_{-1}^{1} (1 - t^2) \times 3 \, \mathrm{d}t \\ &= \int_{-1}^{1} (3 - 3t^2) \, \mathrm{d}t \\ &= \left[3t - t^3\right]_{-1}^{1} \\ &= (3 - 1) - (-3 + 1) \\ &= 4 \end{aligned}$$

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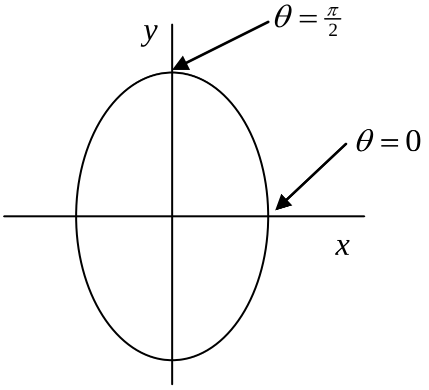
You often need to use trigonometric identities when finding an area from parametric equations, since these often involve trig functions. This is shown in the next example.

Example 5
Find the area of the ellipse given by the parametric equations
$$x = 2\cos\theta, \quad y = 3\sin\theta.$$

Solution



The area of the whole ellipse is four times the area of the part of the curve from $\theta = \frac{\pi}{2}$ to $\theta = 0$.

$$\frac{dx}{d\theta} = -2\sin\theta$$

$$\begin{aligned}\text{Area } = 4\int_{\pi/2}^{0} y\frac{dx}{d\theta}\,d\theta &= 4\int_{\pi/2}^{0}(3\sin\theta)(-2\sin\theta)\,d\theta \\ &= -24\int_{\pi/2}^{0}\sin^2\theta\,d\theta \\ &= -24\int_{\pi/2}^{0}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta \\ &= -12\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/2}^{0} \\ &= -12\left(0 - \tfrac{\pi}{2}\right) \\ &= 6\pi\end{aligned}$$

Using the identity
$\cos 2\theta = 1 - 2\sin^2\theta$