Section 3: The product and quotient rules

Notes and Examples

These notes contain subsections on
- The product rule
- The quotient rule

The product rule

Suppose you want to differentiate $y = x^2\sqrt{1+x}$. In this case, the function y is the product of two simpler functions, $u = x^2$ and $v = \sqrt{1+x}$. You can differentiate u and v:

$$\frac{du}{dx} = 2x, \quad \frac{dv}{dx} = \tfrac{1}{2}(1+x)^{-\frac{1}{2}}$$

You might think that the derivative of y is simply the product of the derivatives of u and v. In fact, the formula is a bit more complicated:

If $y = u \times v$, then: $\dfrac{dy}{dx} = u\dfrac{dv}{dx} + v\dfrac{du}{dx}$.

This is called the **product rule**.

Applying this formula to the example above:

$$\begin{aligned}
&y = x^2\sqrt{1+x} \\
&u = x^2 \Rightarrow \frac{du}{dx} = 2x \\
&v = \sqrt{1+x} \Rightarrow \frac{dv}{dx} = \tfrac{1}{2}(1+x)^{-\frac{1}{2}} \\
&\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \\
&\quad = x^2 \times \tfrac{1}{2}(1+x)^{-\frac{1}{2}} + (1+x)^{\frac{1}{2}} \times 2x
\end{aligned}$$

The tricky part of questions like this is to simplify your answer. In this case, you can take out a factor of $\tfrac{1}{2}x(1+x)^{-\frac{1}{2}}$:

$$\begin{aligned}
\frac{dy}{dx} &= \tfrac{1}{2}x^2(1+x)^{-\frac{1}{2}} + 2x(1+x)^{\frac{1}{2}} \\
&= \tfrac{1}{2}x(1+x)^{-\frac{1}{2}}\left(x + 4(1+x)\right) \\
&= \tfrac{1}{2}x(1+x)^{-\frac{1}{2}}(5x+4)
\end{aligned}$$

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Example 1

Given that $y = x^3(1+3x)^{\frac{1}{3}}$, show that $\frac{dy}{dx} = x^2(1+3x)^{-\frac{2}{3}}(3+10x)$. Hence find the turning points of this curve.

Solution

$y = uv$ where $u = x^3, v = (1+3x)^{\frac{1}{3}}$

$$u = x^3 \Rightarrow \frac{du}{dx} = 3x^2$$
$$v = (1+3x)^{\frac{1}{3}} \Rightarrow \frac{dv}{dx} = \tfrac{1}{3}(1+3x)^{-\frac{2}{3}} \times 3 = (1+3x)^{-\frac{2}{3}}$$

using the chain rule

Using the product rule:

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= x^3(1+3x)^{-\frac{2}{3}} + (1+3x)^{\frac{1}{3}} \times 3x^2$$
$$= x^2(1+3x)^{-\frac{2}{3}}(x+3+9x)$$
$$= x^2(1+3x)^{-\frac{2}{3}}(3+10x) \text{ as required.}$$

Take out $x^2(1+3x)^{-\frac{2}{3}}$ as a factor

The turning points occur when $\frac{dy}{dx} = 0$

$x^2(1+3x)^{-\frac{2}{3}}(3+10x) = 0$

$\Rightarrow x = 0$ or $x = -\frac{3}{10}$.

When $x = 0$, $y = 0$

When $x = -0.3$, $y = -0.0125$

So the turning points are (0, 0) and (–0.3, –0.0125).

The quotient rule

You have now learnt a technique for differentiating composite functions (the chain rule) and a technique for differentiating products. The third differentiation technique deals with quotients.

For a function y which can be expressed as a quotient of two simpler functions u and v, there is a formula for the derivative of y:

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \dfrac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

This is called the **quotient rule**. You can prove the quotient rule by thinking of the quotient $\frac{u}{v}$ as the product $u \times v^{-1}$ and applying the product rule.

Example 2

Differentiate $y = \dfrac{x^2}{1+2x^3}$, simplifying your answer. Hence find the turning points of this curve and determine their nature.

Solution

$y = \dfrac{u}{v}$ where $u = x^2$ and $v = 1 + 2x^3$

$u = x^2 \Rightarrow \dfrac{du}{dx} = 2x$

$v = 1 + 2x^3 \Rightarrow \dfrac{dv}{dx} = 6x^2$

Using the quotient rule:

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{(1+2x^3)\times 2x - x^2 \times 6x^2}{(1+2x^3)^2}$$

$$= \frac{2x + 4x^4 - 6x^4}{(1+2x^3)^2}$$

$$= \frac{2x - 2x^4}{(1+2x^3)^2}$$

$$= \frac{2x(1-x^3)}{(1+2x^3)^2}$$

It is best to factorise answers as far as possible.

The turning points occur when $\dfrac{dy}{dx} = 0$

$\dfrac{2x(1-x^3)}{(1+2x^3)^2} = 0 \Rightarrow x = 0$ or $x = 1$

When $x = 0$, $y = 0$

When $x = 1$, $y = \frac{1}{3}$.

To investigate the nature of the turning points, you could find $\dfrac{d^2y}{dx^2}$, but this would be complicated. It is easier to investigate the sign of $\dfrac{dy}{dx}$ for values of x before and after the stationary values. This can be done in a table like this:

x	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$\dfrac{dy}{dx}$	– \	0 —	+ /	0 —	– \

So (0, 0) is a minimum and (1, $\frac{1}{3}$) is a maximum.