Section 2: The chain rule

Section test

1. The derivative of $(2x + 1)^4$ is

(a) $4(2x + 1)^3$
(b) $\frac{2}{5}(2x+1)^5$
(c) $8(2x + 1)^3$
(d) $\frac{1}{5}(2x+1)^5$

2. The derivative of $\sqrt{x^2-3}$ is

(a) $\dfrac{x}{2\sqrt{x^2-3}}$
(b) $\dfrac{x}{\sqrt{x^2-3}}$
(c) $\frac{1}{2}(x^2-3)^{-\frac{1}{2}}$
(d) $\frac{4}{3}x(x^2-3)^{\frac{3}{2}}$

3. Find the gradient of the curve $y = \dfrac{1}{1-2x}$ at the point (1, –1).

4. Find the equation of the tangent to the curve $y=(3x-2)^3$ at the point where $x=1$.

5. Find the equation of the normal to the curve $y = \sqrt{1+x^3}$ at the point (2, 3).

6. Find the turning points on the curve $y = \sqrt{2x-x^2}$.

7. The gradient of the curve $y = f(x)$ at the point (1, 2) is 3.
The inverse function of $y = f(x)$ is $y = f^{-1}(x)$.
On the curve $y = f^{-1}(x)$, what is the corresponding point and its gradient?

8. Find the gradient of the curve $x = y+\dfrac{1}{y}$ at the point (2.5, 2).

9. When a light source is r cm from a point O, the light intensity at O is $\dfrac{100}{r^2}$ lux. If $r=5$ and the light source is moving away at a rate of 0.5 cm s^{-1}, find the rate of change of the light intensity.

10. The radius of a spherical balloon is 2 m and its volume is increasing at a rate of 0.1 m^3 per minute. The volume V of a sphere of radius r is $V = \dfrac{4}{3}\pi r^3$. What is the rate at which the radius is increasing?

Section test Solutions

1) Let $u = 2x + 1 \Rightarrow \frac{du}{dx} = 2$

$y = u^4 \Rightarrow \frac{dy}{du} = 4u^3$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 4u^3 \times 2$$
$$= 8u^3$$
$$= 8(2x+1)^3$$

2) Let $u = x^2 - 3 \Rightarrow \frac{du}{dx} = 2x$

$y = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \tfrac{1}{2}u^{-\frac{1}{2}} \times 2x$$
$$= xu^{-\frac{1}{2}}$$
$$= x\left(x^2 - 3\right)^{-\frac{1}{2}}$$
$$= \frac{x}{\sqrt{x^2 - 3}}$$

3) Let $u = 1 - 2x \Rightarrow \frac{du}{dx} = -2$

$y = \frac{1}{u} = u^{-1} \Rightarrow \frac{dy}{du} = -u^{-2} = -\frac{1}{u^2}$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= \left(-\frac{1}{u^2}\right)x(-2) = \frac{2}{(1-2x)^2}$$

The gradient of the curve at the point $x = 1$: $m = \frac{2}{(1-2\times 1)^2} = 2$

4) Let $u = 3x - 2 \Rightarrow \frac{du}{dx} = 3$

$y = u^3 \Rightarrow \frac{dy}{du} = 3u^2$

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Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= 3u^2 \times 3 = 9(3x-2)^2$

The gradient of the curve at the point $x = 1$: $m = 9 \times 1^2 = 9$

When $x = 1, y = 1$

$y - y_1 = m(x - x_1)$

$y - 1 = 9(x - 1)$

$y - 1 = 9x - 9$

$y = 9x - 8$

5) Let $u = 1 + x^3 \Rightarrow \frac{du}{dx} = 3x^2$

$y = \sqrt{u} = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$= \frac{1}{2\sqrt{u}} \times 3x^2$

$= \frac{3x^2}{2\sqrt{u}}$

$= \frac{3x^2}{2\sqrt{1+x^3}}$

The gradient of the curve at the point $x = 2$: $m = \frac{3 \times 2^2}{2\sqrt{1+2^3}} = 2$

Therefore the gradient of the normal is $-\frac{1}{2}$.

$y - y_1 = m(x - x_1)$

$y - 3 = -\frac{1}{2}(x - 2)$

$2(y - 3) = -(x - 2)$

$2y - 6 = -x + 2$

$2y = -x + 8$

$y = -\frac{1}{2}x + 4$

6) Let $u = 2x - x^2 \Rightarrow \frac{du}{dx} = 2 - 2x$

$y = \sqrt{u} = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = \frac{1}{2}u^{-\frac{1}{2}} \times (2 - 2x) = \frac{1 - x}{\sqrt{2 - 2x^2}}$

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As we want the turning points, we want to know when $\frac{dy}{dx} = 0$

$$\frac{1-x}{\sqrt{2-2x^2}} = 0$$
$$1 - x = 0$$
$$1 = x$$
$$x = 1$$

In the original equation, when $x = 1$, $y = \sqrt{2(1)-(1)^2} = \sqrt{1} = 1$ only

Therefore, the turning point is $(1,1)$.

7) The graph of the inverse of a function $y = f(x)$ is found by reflecting the graph of $y = f(x)$ in the line $y = x$.
If you reflect the point $(1,2)$ in the line $y = x$, the point becomes $(2,1)$.
The point $(2,1)$ for $y = f^{-1}(x)$ is a reflection of the point $(1,2)$ of $y = f(x)$, therefore the gradient at the point $(2,1)$ is $\frac{1}{3}$.

8) $x = y + \frac{1}{y} = y + y^{-1}$

$$\frac{dx}{dy} = 1 - y^{-2} = 1 - \frac{1}{y^2}$$
$$\frac{dy}{dx} = 1 \Big/ \left(1 - \frac{1}{y^2}\right)$$

When $y = 2$, $= 1 \Big/ \left(1 - \frac{1}{2^2}\right) = 1/\frac{3}{4} = \frac{4}{3}$

9) $I = \frac{100}{r^2}$

$$\frac{dI}{dr} = -\frac{200}{r^3}$$

When $r = 5$, $\frac{dr}{dt} = 0.5$

Using the chain rule: $\frac{dI}{dt} = \frac{dr}{dt} \times \frac{dI}{dr}$

$$= 0.5 \times \left(-\frac{200}{(5)^3}\right)$$
$$= -0.8 \text{ lux.s}^{-1}$$

10) $\frac{dV}{dt} = 0.1$

$$V = \tfrac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dr} = 4\pi r^2 \Rightarrow \frac{dr}{dV} = \frac{1}{4\pi r^2}$$

Using the chain rule: $\frac{dr}{dt} = \frac{dV}{dt} \times \frac{dr}{dV}$

$$= (0.1) \times \left(\frac{1}{4\pi r^2}\right)$$

$$= \frac{1}{40\pi r^2}$$

$$= \frac{1}{40\pi (2)^2}$$

$$= \frac{1}{160\pi}$$

$= 0.00199$m per min