Section 2: The chain rule

Notes and Examples

This chapter extends methods for differentiating functions.

These notes contain subsections on

The chain rule

Suppose you want to differentiate the function $y=(2x+5)^3$. You could expand the brackets:

$$\begin{aligned} y &= (2x+5)^3 \\ &= (2x)^3+3(2x)^2.5+3(2x).5^2+5^3 \\ &= 8x^3+60x^2+150x+125 \end{aligned}$$

Then you could differentiate term by term:

$$\begin{aligned} \frac{dy}{dx} &= (8\times 3x^2)+(60\times 2x)+150 \\ &= 24x^2+120x+150 \end{aligned}$$

This result can be factorised:

$$\begin{aligned} \frac{dy}{dx} &= 24x^2+120x+150 \\ &= 6(4x^2+20x+25) \\ &= 6(2x+5)^2 \end{aligned}$$

This suggests that there is perhaps a more efficient method.
The expression for the derivative can be written as

$$y=(2x+5)^3 \quad \Rightarrow \quad \frac{dy}{dx} = 2\times 3(2x+5)^2$$

The derivative of (...)^3 is 3(...)^2

The derivative of $(2x + 5)$ is 2

To make the method is clearer, let $u = 2x + 5$. Then $y = u^3$

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Now $\frac{dy}{du} = 3u^2$ and $\frac{du}{dx} = 2$

So $\frac{dy}{dx} = 2 \times 3(2x+5)^2 = 3u^2 \times 2 = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 2 \times 3\left(2x+5\right)^2 = 3u^2 \times 2 = \frac{dy}{du} \times \frac{du}{dx}.$$

This is called the **chain rule**: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$.

Example 1
Differentiate $y = (3x^2 - 4)^7$

Solution

Let $u = 3x^2 - 4 \Rightarrow \frac{du}{dx} = 6x$

$y = u^7 \Rightarrow \frac{dy}{du} = 7u^6$

Using the chain rule:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 7u^6 \times 6x \\ &= 42xu^6 \\ &= 42x\left(3x^2 - 4\right)^6 \end{aligned}$$

After some practice, you may find that you can differentiate composite functions like the one in Example 1 without introducing the variable u.

$$y = \left(3x^2 - 4\right)^7 \Rightarrow \frac{dy}{dx} = 7\left(3x^2 - 4\right)^6 \times 6x$$

Example 2
Differentiate $y = \sqrt{1 + x^2}$

Solution

Let $u = 1 + x^2 \Rightarrow \frac{du}{dx} = 2x$

$y = \sqrt{u} = u^{\frac{1}{2}} \Rightarrow \frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \tfrac{1}{2}u^{-\frac{1}{2}} \times 2x$$
$$= \frac{x}{\sqrt{u}}$$
$$= \frac{x}{\sqrt{1+x^2}}$$

Alternatively, differentiate directly:

Differentiate $(\ldots)^{1/2}$ to get ½ $(\ldots)^{-1/2}$…

$$y = \left(1+x^2\right)^{\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \tfrac{1}{2}\left(1+x^2\right)^{-\frac{1}{2}} \times 2x$$
$$= x\left(1+x^2\right)^{-\frac{1}{2}}$$

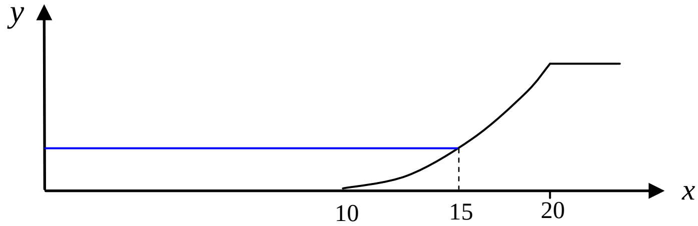
…then multiply by the derivative of $(1 + x^2)$

Rates of change

The chain rule is also used to solve problems involving related rates of change.

Example 3

A bank of a reservoir is modelled by the curve $y = \frac{x^2}{50} - 2$ for $10 \le x \le 20$, where x is measured in metres (see diagram).



The width of the reservoir is 15 m and is increasing at a rate of 0.2 m per day. Calculate the rate of change of the depth of the reservoir.

Solution

$y = \frac{x^2}{50} - 2 \Rightarrow \frac{dy}{dx} = \frac{x}{25}$.

When $x = 15$, $\frac{dy}{dx} = \frac{15}{25} = \frac{3}{5}$

The rate of increase in width is given by $\frac{dx}{dt}$, so $\frac{dx}{dt} = 0.2$.

The rate of change of depth is given by $\frac{dy}{dt}$.

By the chain rule:

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= \frac{3}{5} \times 0.2 \\ &= 0.12\end{aligned}$$

So the rate of change of depth is 0.12 m per day.

Inverse functions

For the function $y = x^3$, $\frac{dy}{dx} = 3x^2$

The function can be rewritten with x as the subject: $x = y^{\frac{1}{3}}$,

Differentiating this gives $\frac{dx}{dy} = \frac{1}{3} y^{-\frac{2}{3}} = \frac{1}{3y^{\frac{2}{3}}} = \frac{1}{3x^2}$.

Try this with other functions.
You should find that your results suggest the relationship

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

This relationship can be useful if you need to differentiate a function in which x is given in terms of y and which cannot easily be rewritten to give y in terms of x. (However, functions like this can also be differentiated using implicit differentiation – this will be covered later in A level Mathematics).