Section 1: Differentiating exponentials and logarithms

Notes and Examples

These notes contain subsections on

- Differentiating the exponential functions
- Differentiating other exponential functions
- Differentiating logarithms

Differentiating exponential functions

You can learn something about the derivative of exponential functions like $y = a^x$ by trying to differentiate using first principles:

Suppose we increase x by a small change δx, resulting in a small change in y δy:

$$y = a^x$$

$$y + \delta y = a^{x+\delta x}$$

Subtracting:
$$\delta y = a^{x+\delta x} - a^x$$
$$= a^x a^{\delta x} - a^x$$
$$= a^x(a^{\delta x} - 1)$$

Dividing by δx:
$$\frac{\delta y}{\delta x} = \frac{a^x(a^{\delta x} - 1)}{\delta x}$$

$\Rightarrow \quad \frac{\mathrm{d}y}{\mathrm{d}x} = ka^x$ where k is the limit of the expression $\frac{a^{\delta x} - 1}{\delta x}$ as $\delta x \to 0$.

Putting $x = 0$ into the expression for $\frac{\delta y}{\delta x}$ tells you that k is actually the gradient of the graph $y = a^x$ at the origin.

It follows that if you can find a value for a so that this gradient is 1 then, for this value of a, $\frac{\mathrm{d}y}{\mathrm{d}x} = a^x$. This value is just the constant e = 2.718…, which is the base of natural logarithms. You can verify this result numerically in your calculator, by finding the value of $\frac{\mathrm{e}^{0.0001} - 1}{0.0001}$.

It follows that:

$$y = \mathrm{e}^x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x$$

This result can be extended to differentiate $y = \mathrm{e}^{kx}$, where k is a constant, using the chain rule:

$$y = e^{kx}$$

$$\text{Let } u = kx \Rightarrow \frac{du}{dx} = k$$

$$y = e^{u} \Rightarrow \frac{dy}{du} = e^{u}$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= ke^{u}$$
$$= ke^{kx}$$

$$y = e^{kx} \Rightarrow \frac{dy}{dx} = ke^{kx}$$

Example 1
Differentiate:

(i) $y = xe^{2x}$ (ii) $y = \dfrac{x^2}{1+2e^{3x}}$ (iii) $y = e^{x^2}$

Solution

(i) Using the product rule with $u = x$ and $v = e^{2x}$

$$u = x \Rightarrow \frac{du}{dx} = 1$$

$$v = e^{2x} \Rightarrow \frac{dv}{dx} = 2e^{2x}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$
$$= x \times 2e^{2x} + 1 \times e^{2x} .$$
$$= (2x+1)e^{2x}$$

(ii) Using the quotient rule with $u = x^2$ and $v = 1 + 2e^{3x}$,

$$u = x^2 \Rightarrow \frac{du}{dx} = 2x$$

$$v = 1 + 2e^{3x} \Rightarrow \frac{dv}{dx} = 6e^{3x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{(1+2e^{3x})2x - x^2 \times 6e^{3x}}{(1+2e^{3x})^2}$$
$$= \frac{2x + 4xe^{3x} - 6x^2e^{3x}}{(1+2e^{3x})^2}$$

(iii) Using the chain rule with $u = x^2$

$$u = x^2 \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = 2x$$

$$y = \mathrm{e}^u \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}u} = \mathrm{e}^u$$

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \mathrm{e}^u \times 2x \\ &= 2x\mathrm{e}^{x^2}\end{aligned}$$

Differentiating other exponential functions

You may also need to differentiate other exponential functions, such as $y = 2^x$.

You can do this by expressing the function in terms of the exponential function:

$$y = 2^x = \mathrm{e}^{\ln 2^x} = \mathrm{e}^{x \ln 2}$$

Then you can use the chain rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \ln 2 \times \mathrm{e}^{x \ln 2} = \ln 2 \times \mathrm{e}^{\ln 2^x} = \ln 2 \times 2^x = 2^x \ln 2$$

More generally:

$$y = a^x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = a^x \ln a$$

Differentiating logarithms

You can now differentiate the inverse function of $y = \mathrm{e}^x$, which is $y = \ln x$:

$$y = \ln x \Rightarrow x = \mathrm{e}^y$$

$$\Rightarrow \frac{\mathrm{d}x}{\mathrm{d}y} = \mathrm{e}^y$$

$$\Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\mathrm{e}^y} = \frac{1}{x}.$$

$$y = \ln x \Rightarrow \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{x}$$

Example 2

Differentiate:

(i) $\sqrt{x} \ln x$ (ii) $\dfrac{\ln x}{1 + \ln x}$ (iii) $\ln(1 + x^3)$

Solution

(i) Using the product rule with $u = \sqrt{x}$ and $v = \ln x$

$$u = \sqrt{x} = x^{\frac{1}{2}} \Rightarrow \frac{du}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$v = \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = v\frac{du}{dx} + u\frac{dv}{dx}$$

$$= \sqrt{x} \times \frac{1}{x} + \frac{1}{2\sqrt{x}}\ln x$$

$$= \frac{1}{\sqrt{x}} + \frac{\ln x}{2\sqrt{x}}$$

$$= \frac{2 + \ln x}{2\sqrt{x}}$$

(ii) Using the quotient rule with $u = \ln x$ and $v = 1 + \ln x$

$$u = \ln x \Rightarrow \frac{du}{dx} = \frac{1}{x}$$

$$v = 1 + \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1 + \ln x) \times \frac{1}{x} - \ln x \times \frac{1}{x}}{(1 + \ln x)^2}$$

$$= \frac{1 + \ln x - \ln x}{x(1 + \ln x)^2}$$

$$= \frac{1}{x(1 + \ln x)^2}$$

(iii) Using the chain rule with $u = 1 + x^3$

$$u = 1 + x^3 \Rightarrow \frac{du}{dx} = 3x^2$$

$$y = \ln u \Rightarrow \frac{dy}{du} = \frac{1}{u}$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= \frac{1}{u} \times 3x^2$$

$$= \frac{3x^2}{(1 + x^3)}$$

Using logarithms gives an alternative approach to finding the derivative of $y = a^x$.

You can take logarithms of both sides:

$$\ln y = \ln a^x = x \ln a$$

Then rewrite as

$$x = \frac{\ln y}{\ln a}$$

and differentiate x with respect to y.

$$\frac{dx}{dy} = \frac{1}{y \ln a}$$

So $\quad \dfrac{dy}{dx} = y \ln a = a^x \ln a$