Section 2: Differentiating trigonometric functions

Notes and Examples

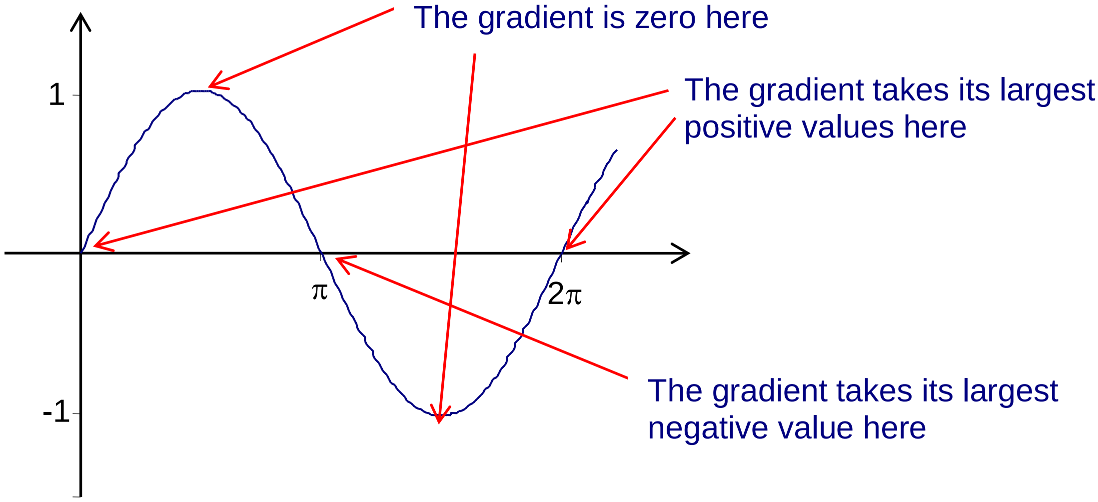
These notes contain subsections on

- The derivative of sin x
- The derivative of cos x
- The derivative of tan x
- Differentiating functions involving sine, cosine and tangent

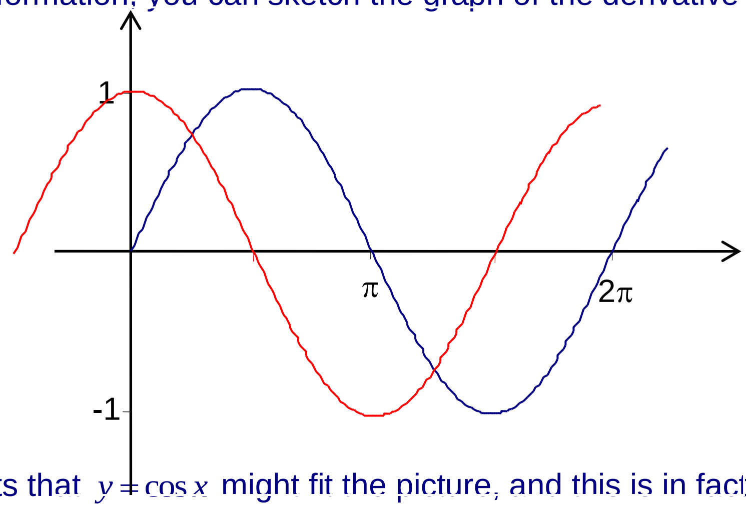
The derivative of sin *x*

What does the derivative of $y = \sin x$ look like?

- It is clearly a periodic function, with period 2π, as the values of the gradient of the function must repeat itself every 2π just as y does;



Using this information, you can sketch the graph of the derivative of sin x:



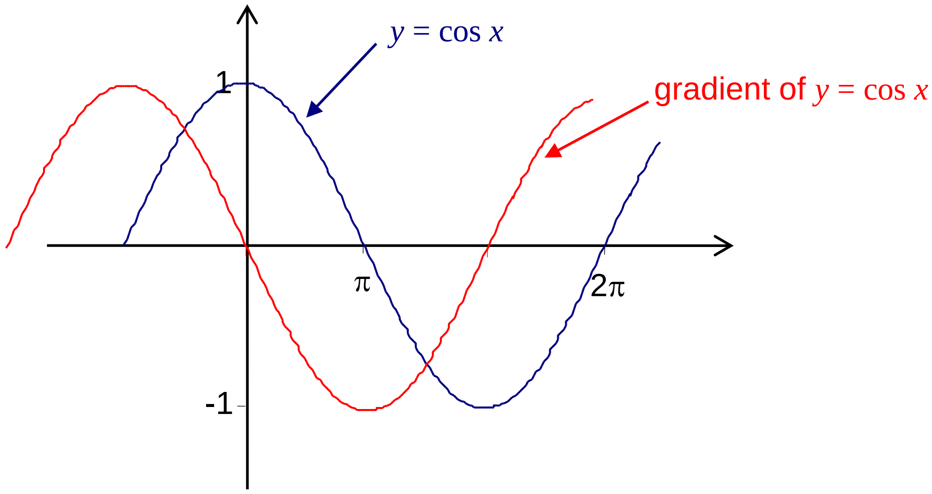
This suggests that $y = \cos x$ might fit the picture, and this is in fact the case, although you need to know more about trig functions to prove this.

IMPORTANT

Note: It is important to remember that this result is only true if x is measured in radians! If you are measuring in degrees, the gradient will not be between -1 and 1.

The derivative of cos *x*

What about the derivative of cos x? The graph of cos x is that of sin x translated $\frac{\pi}{2}$ to the left, so the gradient function will be that of sin x translated $\frac{\pi}{2}$ to the left:



This looks like the reflection of $y = \sin x$ in the x-axis, which is $y = -\sin x$. So:

$$\frac{\mathrm{d}}{\mathrm{d}x}(\sin x) = \cos x \,, \quad \frac{\mathrm{d}}{\mathrm{d}x}(\cos x) = -\sin x$$

The derivative of tan *x*

What about the derivative of tan x?
You can do this by applying the quotient rule to trig functions.

Example 1
Using the derivatives of sin x and cos x, prove that the derivative of tan x is $\sec^2 x$.

Solution
You know that $\tan x = \dfrac{\sin x}{\cos x}$.

Using the quotient rule with $u = \sin x$ and $v = \cos x$

$$u = \sin x \Rightarrow \frac{\mathrm{d}u}{\mathrm{d}x} = \cos x$$

$$v = \cos x \Rightarrow \frac{\mathrm{d}v}{\mathrm{d}x} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$
$$= \frac{\cos x \times \cos x - \sin x \times (-\sin x)}{(\cos x)^2}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x}$$
$$= \sec^2 x$$

since $\cos^2 x + \sin^2 x = 1$

This is another result which you should remember and may be quoted.

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Differentiating functions involving sine, cosine and tangent

You can now apply all the differentiation techniques you have learned to functions involving sines, cosines and tangents.

Functions of the form $y = \sin kx$, $y = \cos kx$ and $y = \tan kx$, where k is a constant, can be differentiated using the chain rule.

Example 2
Differentiate $y = \sin 2x$

Solution
Using the chain rule with $u = 2x$:

$$u = 2x \Rightarrow \frac{du}{dx} = 2$$
$$y = \sin u \Rightarrow \frac{dy}{du} = \cos u$$
$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= \cos u \times 2$$
$$= 2\cos 2x$$

The result from Example 2 can be generalised:

$$\frac{d}{dx}(\sin kx) = k\cos kx, \quad \frac{d}{dx}(\cos kx) = -k\sin kx, \quad \frac{d}{dx}(\tan kx) = k\sec^2 kx$$

You can use these results directly, without having to write out the chain rule.

Here are some more examples.

Example 3

Differentiate:

(i) $\cos^2 x$

(ii) $\sin x^\circ$

(iii) $e^x \tan x$

(iv) $\dfrac{\sin x}{1+\cos x}$,

Solution

(i) Using the chain rule:

$$u = \cos x \Rightarrow \frac{du}{dx} = -\sin x$$

$$y = u^2 \Rightarrow \frac{dy}{du} = 2u$$

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$
$$= 2u \times (-\sin x)$$
$$= -2\cos x \sin x$$

You can only differentiate trig functions when they are measured in radians, so convert to radians by multiplying by $\dfrac{\pi}{180}$

(ii) $$y = \sin x^\circ = \sin\left(\frac{\pi x}{180}\right)$$

$$\frac{dy}{dx} = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right)$$
$$= \frac{\pi}{180}\cos x^\circ$$

Using the standard result for differentiating $\cos kx$

(iii) Using the product rule:

$$u = e^x \Rightarrow \frac{du}{dx} = e^x$$

$$v = \tan x \Rightarrow \frac{dv}{dx} = \sec^2 x$$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= e^x \sec^2 x + e^x \tan x$$

(iv) Using the quotient rule:

$$u = \sin x \Rightarrow \frac{du}{dx} = \cos x$$

$$v = 1 + \cos x \Rightarrow \frac{dv}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

$$= \frac{(1+\cos x)\cos x - \sin x(-\sin x)}{(1+\cos x)^2}$$

$$= \frac{\cos x + \cos^2 x + \sin^2 x}{(1+\cos x)^2}$$

$$= \frac{\cos x + 1}{(1+\cos x)^2}$$

$$= \frac{1}{1+\cos x}$$

Notice the use of trigonometric identities to simplify results.