Section 1: Using parametric equations

Notes and Examples

These notes contain subsections on

- The definition of a parametric equation
- Sketching a parametric curve
- Finding the cartesian equation of a curve
- The parametric equation of a circle

The definition of a parametric equation

An equation like $y = 5x + 1$ or $y = 3\sin x + 4\cos x$ or $x^2 + y^2 = 1$ is called a **cartesian equation**. A cartesian equation gives a direct relationship between x and y.
In **parametric equations** x and y are both defined in terms of a third variable (**parameter**) usually t or θ.

For example $x = t$, $y = t^2$ are a pair of **parametric equations**

and $x = \cos\theta$, $y = \sin\theta$ are also a pair of **parametric equations**.

Parametric equations can be used for a complicated curve which doesn't have a simple Cartesian equation.

Sketching a parametric curve

To sketch a curve given its parametric equations follow these steps.

Step 1 Make a table like this one:

t or θ				
x				
y				

Step 2 Choose values of t or θ (these will be usually be given to you)

Step 3 Work out the corresponding values of x and y using the parametric equations.

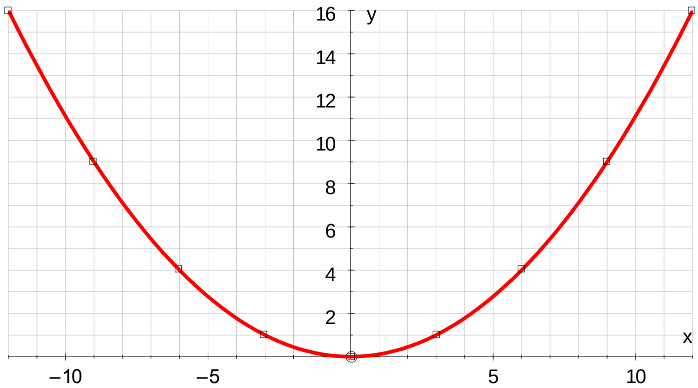
Step 4 Plot the (x, y) coordinates.
Join them up in a smooth curve.

These examples shows you how to do this.

Example 1
A curve has the parametric equations $x = 3t, y = t^2$.
Sketch the curve for $-4 \le t \le 4$

Solution

t	-4	-3	-2	-1	0	1	2	3	4
x	-12	-9	-6	-3	0	3	6	9	12
y	16	9	4	1	0	1	4	9	16



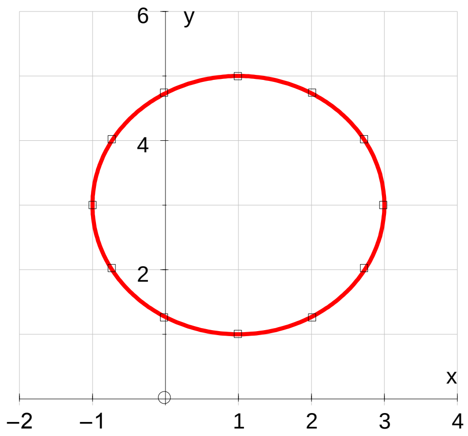
Example 2

A curve has the parametric equations $x = 1 + 2\cos\theta, y = 3 + 2\sin\theta$.

Sketch the curve for $0° \leq \theta \leq 360°$

Solution

θ	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°	300°	330°	360°
x	3	2.73	2	1	0	-0.73	-1	-0.73	0	1	2	2.73	3
y	3	4	4.73	5	4.73	4	3	2	1.27	1	1.27	2	3



Finding the cartesian equation of a curve

To find the cartesian equation of a curve from its parametric equations you need to eliminate the parameter t or θ.

It is not always possible to find the Cartesian equation of a curve defined parametrically.

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There are essentially 3 methods which you will need to use depending on the parametric equations.

Method 1
Use this method when it is straightforward to make t the subject of one equation.

Step 1 Make t the subject of one of the parametric equations.
Step 2 Substitute your equation for t into the other parametric equation.
Step 3 Simplify.

Example 3
Find the cartesian equation of the curve defined by the parametric equations
$x = 3t + 2,\ y = 1 - t^2$

Solution

Step 1 Make t the subject of $x = 3t + 2$: $t = \dfrac{x-2}{3}$ ①

Step 2 Substitute ① into $y = 1 - t^2$: $y = 1 - \left(\dfrac{x-2}{3}\right)^2$

Step 3 Simplify:

$$y = 1 - \left(\frac{x-2}{3}\right)^2$$

$$y = 1 - \frac{(x-2)^2}{9}$$

Sometimes it is easier to rearrange both equations to give an expression for t, as shown in the next example.

Example 4
Find the cartesian equation of the curve defined by the parametric equations
$x = \dfrac{t}{t+1},\ y = \dfrac{t}{t-1}$.

Solution

$$x = \frac{t}{t+1} \Rightarrow x(t+1) = t$$
$$\Rightarrow tx + x = t$$
$$\Rightarrow t - tx = x$$
$$\Rightarrow t(1-x) = x$$
$$\Rightarrow t = \frac{x}{1-x}$$

$$y = \frac{t}{t-1} \Rightarrow y(t-1) = t$$
$$\Rightarrow ty - y = t$$
$$\Rightarrow ty - t = y$$
$$\Rightarrow t(y-1) = y$$
$$\Rightarrow t = \frac{y}{y-1}$$

Equating the two expressions for t:

$$\frac{x}{1-x} = \frac{y}{y-1}$$
$$x(y-1) = y(1-x)$$
$$xy - x = y - xy$$
$$2xy - y = x$$
$$y(2x-1) = x$$
$$y = \frac{x}{2x-1}$$

Method 2

Use this method when it is **not** straightforward to make t the subject of one equation.

Step 1 Check whether adding/subtracting the two equations will result in a 3rd equation in which t can easily be made the subject.
If not, look to rearrange the equations (e.g. squaring, cross multiplying) so this can be done.

Step 2 Then proceed as method 1.

Method 3

Use this method when you have trigonometric functions in the parametric equations.

Step 1 Find an identity which connects the two trigonometric functions.

Step 2 Rearrange the parametric equations so that you can substitute them into the trig identity.

Step 3 Simplify.

Example 5

Find the cartesian equation of the curve defined by the parametric equations
$x = 4 + \cos\theta, \quad y = 2\cos 2\theta$

Solution

The identity which connects $\cos 2\theta$ and $\cos\theta$ is

$$\cos 2\theta \equiv 2\cos^2\theta - 1 \quad ①$$

$$x = 4 + \cos\theta \Rightarrow \cos\theta = x - 4 \quad ②$$

$$y = 2\cos 2\theta \Rightarrow \cos 2\theta = \frac{y}{2} \quad ③$$

Substitute ② and ③ into ①

$$\frac{y}{2} \equiv 2(x-4)^2 - 1$$

Simplify $\quad y \equiv 4(x-4)^2 - 2$

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The parametric equation of a circle

A circle with radius r and centre $(0, 0)$ has parametric equations:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

A circle with radius r and centre (a, b) has parametric equations:

$$x = a + r\cos\theta$$
$$y = b + r\sin\theta$$

Example 6

(i) Find the cartesian equation of the curve given by the parametric equations

$$x = 2 + 3\cos\theta$$
$$y = 3\sin\theta - 1$$

(ii) Sketch the curve.

Solution

(i) The trig identity $\sin^2\theta + \cos^2\theta \equiv 1$ links $\sin\theta$ and $\cos\theta$.

Make $\cos\theta$ the subject of $x = 2 + 3\cos\theta$: $\cos\theta = \frac{x-2}{3}$ ①

Make $\sin\theta$ the subject of $y = 3\sin\theta - 1$: $\sin\theta = \frac{y+1}{3}$ ②

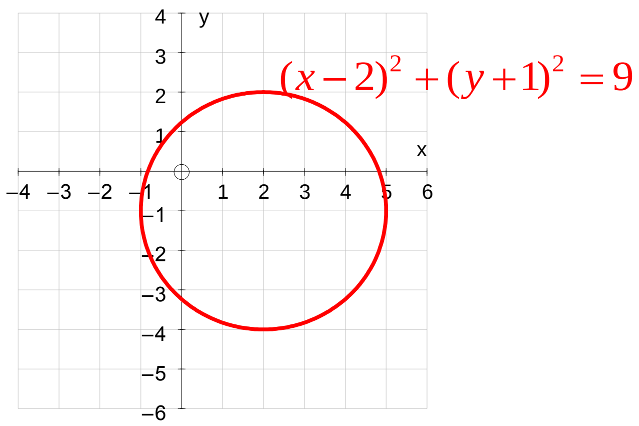
Substitute ① and ② into $\sin^2\theta + \cos^2\theta \equiv 1$:

$$\left(\frac{x-2}{3}\right)^2 + \left(\frac{y+1}{3}\right)^2 = 1$$

So $$\frac{(x-2)^2}{9} + \frac{(y+1)^2}{9} = 1$$

So $$(x-2)^2 + (y+1)^2 = 9$$

(ii) Need to sketch a circle of radius 3 and centre (2, -1)



Example 7

(i) Write down the parametric equations of the curve $x^2 + y^2 = 16$

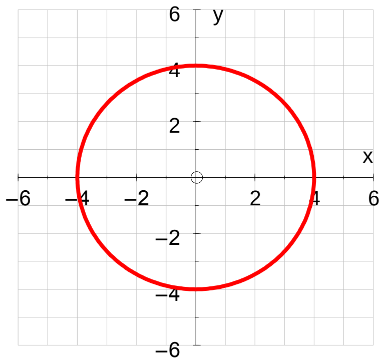
(ii) Sketch the curve.

Solution

(i) This is a circle radius 4 and centre (0, 0)
So the parametric equations are

$$x = 4\cos\theta$$
$$y = 4\sin\theta$$

(ii)



$x = 4\cos\theta$
$y = 4\sin\theta$