Edexcel A level Maths Trigonometric identities

Section 2: Further trigonometric equations

Notes and Examples

In this section you learn to convert expressions in the form $a\sin\theta + b\cos\theta$ to the form $r\cos(\theta \pm \alpha)$ or $r\sin(\theta \pm \alpha)$. This is a useful skill as it enables you to solve equations and sketch curves that you previously wouldn't have been able to.

These notes contain subsections on

- The forms r cos(θ ± α) and r sin(θ ± α)
- Solving equations

The forms *r* cos(*θ* ± *α*) and *r* sin(*θ* ± *α*)

Try using graphing software or a graphical calculator to sketch graphs of the form $y = a\sin\theta + b\cos\theta$, with various values of a and b.

You should find that all the graphs of this form are the same shape as a sine or cosine graph, but translated by various amounts in the x direction, and stretched by various amounts in the y direction.

This suggests that any expression of the form $a\sin\theta + b\cos\theta$ may be written in the form $r\sin(\theta \pm \alpha)$ or $r\cos(\theta \pm \alpha)$ for particular values of r and α.

The compound angle formulae can be used to do this. The results can be summarised as follows:

$$\left.\begin{aligned} a\sin\theta + b\cos\theta &= r\sin(\theta+\alpha) \\ a\sin\theta - b\cos\theta &= r\sin(\theta-\alpha) \\ a\cos\theta + b\sin\theta &= r\cos(\theta-\alpha) \\ a\cos\theta - b\sin\theta &= r\cos(\theta+\alpha) \end{aligned}\right\} \text{where } r = \sqrt{a^2+b^2},\ \cos\alpha = \frac{a}{r} \text{ and } \sin\alpha = \frac{b}{r}$$

However, it is best not to try to learn and apply the formula above as it is easy to get muddled. A better approach is to use the compound angle formulae and then compare coefficients, as shown in Example 1.

Example 1

(i) Find the positive value of r and the acute angle α for which

$$3\sin x + 4\cos x = r\sin(x+\alpha)$$

(ii) Sketch the curve with the equation $y = 3\sin x + 4\cos x$.

Solution

(i) $3\sin x + 4\cos x = r\sin(x+\alpha)$

$= r(\sin x\cos\alpha + \cos x\sin\alpha)$

$= r\sin x\cos\alpha + r\cos x\sin\alpha$

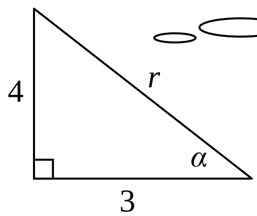
Compare coefficients of cos *x* and sin *x*

$3\sin x + 4\cos x = r\sin x\cos\alpha + r\cos x\sin\alpha$

$\Rightarrow \quad r\cos\alpha = 3$ and $r\sin\alpha = 4$

$\Rightarrow \quad \cos\alpha = \frac{3}{r}$ and $\sin\alpha = \frac{4}{r}$

This information can be illustrated on a right-angled triangle, which allows you to find the values of *r* and α.



$r = \sqrt{3^2 + 4^2} = 5$ and $\tan\alpha = \frac{4}{3} \Rightarrow \alpha = 53.1°$

So: $3\sin x + 4\cos x = 5\sin(x + 53.1°)$

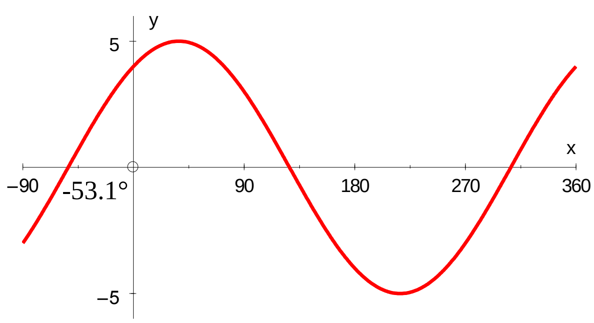
(ii) $y = 3\sin x + 4\cos x \Rightarrow y = 5\sin(x+53.1°)$

The minimum value that $5\sin(x+53.1°)$ has is –5.

The maximum value that $5\sin(x+53.1°)$ has is 5.

Remember $y = \sin\theta$ lies between –1 and 1

The graph of $y = 5\sin(x+53.1°)$ is a translation of the graph $y = \sin x$ by the vector $\begin{pmatrix} -53.1 \\ 0 \end{pmatrix}$ and a one-way stretch parallel to the *y* axis scale factor 5.



Solving equations

The next example shows how this form can be used to solve equations of the form $a\cos\theta + b\sin\theta = c$.

Example 2
Solve the equation $3\sin x + 4\cos x = 3$ for $0° \le x \le 360°$

Solution
From Example 1, you can write $3\sin x + 4\cos x$ as $y = 5\sin(x + 53.1°)$

$$3\sin x + 4\cos x = 3 \Rightarrow 5\sin(x+53.1) = 3$$
$$\Rightarrow \sin(x+53.1) = 0.6$$
$$\Rightarrow x + 53.1 = 36.9° \text{ or } 143.1°$$
$$\Rightarrow x = -16.3° \text{ or } 90°$$

You need to solve $\sin\theta = 0.6$ and then subtract 53.1° from each solution.

$x = -16.3°$ is outside the range, so add 360° to give a solution of $x = 343.7°$

The solutions are x = 90° or 343.7 to 1 d.p.