Section 1: The compound angle identities

Notes and Examples

These notes contain subsections on

- The compound angle formulae
- The double angle formulae

The compound angle formulae

Some useful identities are the compound angle formulae:

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B} \quad (A+B) \neq 90°, 270°, \ldots$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B} \quad (A-B) \neq 90°, 270°, \ldots$$

Remember $\tan\theta$ is undefined for these values.

Example 1

Write $\cos 75°$ as a surd.

Solution

$\cos 75° = \cos(45° + 30°)$

Using the compound-angle formula $\cos(A+B) = \cos A\cos B - \sin A\sin B$ we have:

$$\cos(45° + 30°) = \cos 45°\cos 30° - \sin 45°\sin 30°$$
$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$
$$= \frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}}$$

$\cos 45° = \frac{1}{\sqrt{2}}$ $\quad$ $\sin 45° = \frac{1}{\sqrt{2}}$

$\cos 30° = \frac{\sqrt{3}}{2}$ $\quad$ $\sin 30° = \frac{1}{2}$

The compound angle formulae can also be used to solve equations.

Example 2

Solve $\sin(60° - \theta) = \cos\theta$ for $0° \leq \theta \leq 360°$

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Solution

$\sin(60° - \theta) = \cos\theta$

$\sin 60° \cos\theta - \cos 60° \sin\theta = \cos\theta$

$\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta = \cos\theta$

Use the compound-angle formula $\sin(A - B) = \sin A \cos B - \cos A \sin B$

Remember you can only solve an equation which has either tan θ, cos θ or sin θ as the subject.

Rearrange to collect like terms: $\frac{\sqrt{3}}{2}\cos\theta - \cos\theta = \frac{1}{2}\sin\theta$

Factorise the LHS: $\left(\frac{\sqrt{3}}{2} - 1\right)\cos\theta = \frac{1}{2}\sin\theta$

Now you can divide by $\cos\theta$: $\frac{\sqrt{3}}{2} - 1 = \frac{1}{2}\frac{\sin\theta}{\cos\theta}$

Whenever you have an equation which is a mixture of sines and cosines see if you can use the identity $\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$

Make $\tan\theta$ the subject: $\tan\theta = 2\left(\frac{\sqrt{3}}{2} - 1\right)$

$\Rightarrow \theta = -15°$

So $\theta = 165°$ or $345°$

To find all the solutions in range you need to add on multiples of 180°.

The double angle formulae

In the case where $A = B$, the compound angle formulae become the double angle formulae.

$$\sin 2A = 2\sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$= 1 - 2\sin^2 A$$

$$= 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A} \quad A \neq 45°, 135°, \ldots$$

These three expressions for cos 2*A* are all equivalent

Remember tan 2*A* is undefined for these values.

Example 3

Use the double angle identities to show that $\frac{\sin 2\theta}{\cos 2\theta} \equiv \tan 2\theta$.

Solution

You are aiming for: $\tan 2\theta \equiv \frac{2\tan\theta}{1 - \tan^2\theta}$

Working with the LHS:

Only work with one side of the identity.

$$\frac{\sin 2\theta}{\cos 2\theta} \equiv \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}$$

$$\equiv \frac{\dfrac{2\sin\theta\cancel{\cos\theta}}{\cos^{\cancel{2}}\theta}}{\dfrac{\cancel{\cos^2\theta}}{\cancel{\cos^2\theta}} - \dfrac{\sin^2\theta}{\cos^2\theta}}$$

$$\equiv \frac{\dfrac{2\sin\theta}{\cos\theta}}{1 - \dfrac{\sin^2\theta}{\cos^2\theta}}$$

$$\equiv \frac{2\tan\theta}{1-\tan^2\theta}$$

$$\equiv \tan 2\theta$$

In order to get '$1-\tan^2\theta$' in the denominator, divide both the numerator and the denominator by $\cos^2\theta$:

Remember so long as you multiply both the 'top' and the' bottom' of a fraction by the same thing you don't change it.

So $\dfrac{\sin 2\theta}{\cos 2\theta} \equiv \tan 2\theta$ as required.

In this example you need to use a double-angle formula to solve an equation giving your answer in radians.

Example 4

Solve $\cos 2\theta - 2 = 5\cos\theta$ for $0 \le \theta \le 2\pi$

Look to see which of the three formula for $\cos 2\theta$ is best. This one means that you end up with an equation just in terms of $\cos\theta$.

Solution

Use the double-angle identity $\cos 2A \equiv 2\cos^2 A - 1$

Substituting: $2\cos^2\theta - 1 - 2 = 5\cos\theta$

$2\cos^2\theta - 3 = 5\cos\theta$

This is a quadratic in $\cos\theta$.

Rearranging: $2\cos^2\theta - 5\cos\theta - 3 = 0$

Factorising: $(2\cos\theta + 1)(\cos\theta - 3) = 0$

$\cos\theta = -\tfrac{1}{2}$ or $\cos\theta = 3$

$\cos\theta = 3$ has no solutions

$\cos\theta = -\tfrac{1}{2} \Rightarrow \theta = \dfrac{2\pi}{3}$ or $\theta = \dfrac{4\pi}{3}$

The second solution is found by subtracting from 2π.