Section 1: Trigonometric functions and identities

Notes and Examples

These notes contain subsections on

- **The reciprocal trigonometric functions**
- **The inverse trigonometric functions**
- **Trigonometric identities**

The reciprocal trigonometric functions

In addition to using the sine, cosine and tangent functions you will need to know three more functions. These are the reciprocal trigonometric functions and are defined as:

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} \qquad \sin\theta \neq 0$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \cos\theta \neq 0$$

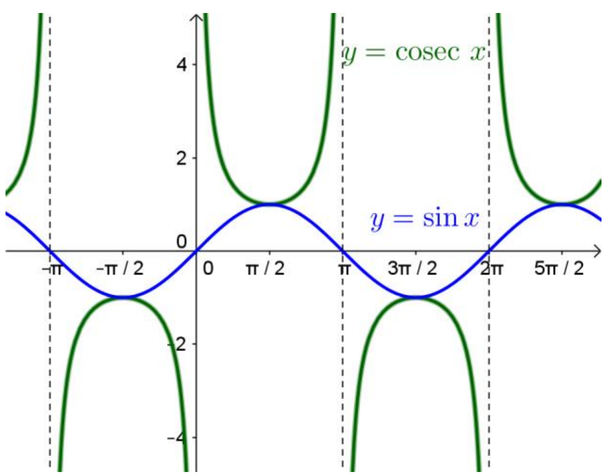
$$\cot\theta = \frac{1}{\tan\theta} \qquad \tan\theta \neq 0$$

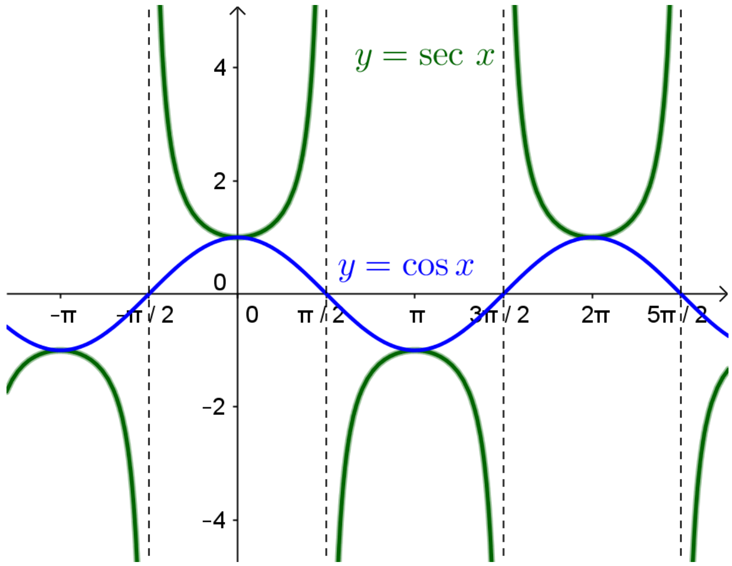
Remember the reciprocal of a number or function is '1 divided by that number or function'

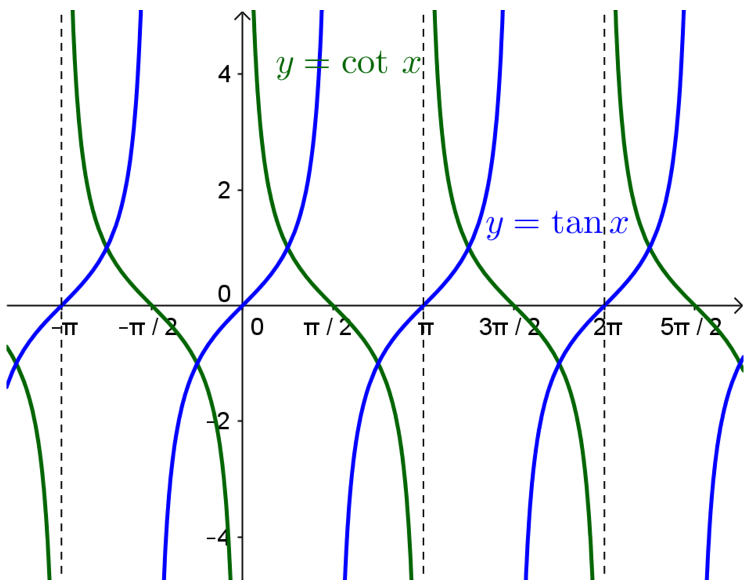
Use the 3rd letter of each function as a memory aid:
e.g. 3rd letter of 'cosec' is s and

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

You need to be familiar with the graphs of all these functions, and how they relate to the graphs of the sin, cos and tan functions.







Example 1
Write down the exact value of cot 120°.

Solution

$\cot 120° = \dfrac{1}{\tan 120°}$

$y = \tan\theta$ has period of 180° so tan 120° = tan (–60°).

$y = \tan\theta$ has rotational symmetry about the origin so tan (–60°) = –tan 60° $= -\sqrt{3}$

So $\cot 120° = -\dfrac{1}{\sqrt{3}}$

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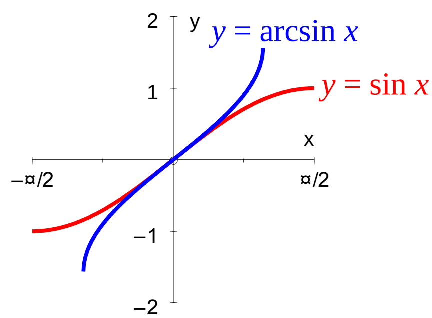
The inverse trigonometric functions

You have already learnt in your work on functions that the graphs of a function and its inverse are reflections of each other in the line $y = x$.

The trigonometric functions $\sin x$, $\cos x$ and $\tan x$ are many-one functions, i.e. for any particular output of these functions, there is more than one input (in fact, for these functions there are an infinitely large number of inputs). For example, there are an infinite number of values of x for which $\sin x = 0.5$.

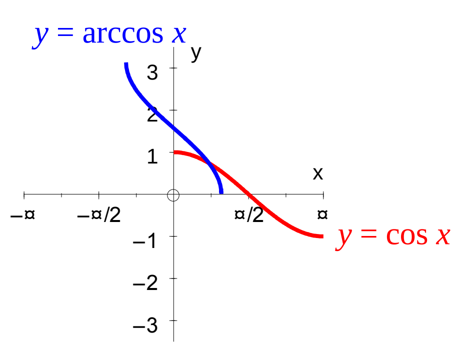
It is therefore not possible to find inverse functions for the trigonometric functions. However, it is possible to find inverse functions if the domains are restricted so that the functions are one-one. These inverse functions are $\arcsin x$, $\arccos x$ and $\arctan x$. In some textbooks these functions are written as $\sin^{-1}x$, $\cos^{-1}x$ and $\tan^{-1}x$.

In the case of $\sin x$, the restricted domain used is $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$.



The domain of the function $y = \arcsin x$ is $-1 \le x \le 1$, and its range is $-\frac{1}{2}\pi \le y \le \frac{1}{2}\pi$

In the case of $\cos x$, a different restricted domain is needed, since the domain $-\frac{1}{2}\pi \le x \le \frac{1}{2}\pi$ does not cover the whole range of $\cos x$. The restricted domain used is $0 \le x \le \pi$.



The domain of the function $y = \arccos x$ is $-1 \le x \le 1$, and its range is $0 \le y \le \pi$.

In the case of $\tan x$, the restricted domain used is again $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$ ($\tan x$ is not defined for $x = \pm\frac{1}{2}\pi$).

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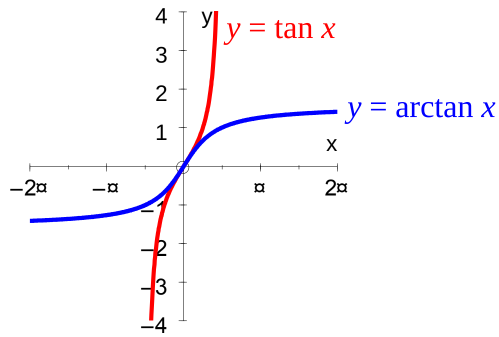


The domain of the function $y = \arctan x$ is $x \in \mathbb{R}$, and its range is $-\frac{1}{2}\pi < y < \frac{1}{2}\pi$.

Example 2
Find the values of
(i) $\cos(\arctan(-1))$
(ii) $\arcsin(-\frac{1}{2}) + \arccos(-\frac{1}{2})$

Solution
(i) $\arctan(-1) = -\frac{\pi}{4}$

$-\frac{1}{2}\pi < \arctan x < \frac{1}{2}\pi$

$\cos(\arctan(-1)) = \cos\left(-\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

(ii) $\arcsin(-\frac{1}{2}) + \arccos(-\frac{1}{2}) = -\frac{\pi}{6} + \frac{2\pi}{3}$

$= \frac{\pi}{2}$

$-\frac{1}{2}\pi \leq \arcsin x \leq \frac{1}{2}\pi$
$0 \leq \arccos x \leq \pi$

Trigonometric identities

Remember: An identity is true for all values of θ.

You already know the following trigonometric identities:

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\sin^2\theta + \cos^2\theta \equiv 1$$

e.g. $\tan 30° = \frac{\sin 30°}{\cos 30°}$

By dividing the identity $\sin^2\theta + \cos^2\theta \equiv 1$ through by $\cos^2\theta$ and by $\sin^2\theta$ two further identities are obtained:

$$\tan^2\theta + 1 \equiv \sec^2\theta$$

$$1 + \cot^2\theta \equiv \mathrm{cosec}^2\theta$$

These two new identities, together with $\sin^2\theta + \cos^2\theta \equiv 1$, are often called the **Pythagorean identities** since they are derived using Pythagoras' theorem.

You need to learn these identities. However, if you are not sure if you are remembering them correctly, then it is easy to check by dividing the identity $\sin^2\theta + \cos^2\theta \equiv 1$ through by $\cos^2\theta$ or by
$\sin^2\theta$ as required.

Example 3
In ΔABC, angle A = 90° and cosec B = 2.
(i) Find angles B and C.
(ii) Find tan C and sec C
(iii) Show that $\tan^2 C + 1 = \sec^2 C$

Solution

(i) $\operatorname{cosec} B = 2 \Rightarrow \sin B = \frac{1}{2}$

$\Rightarrow B = 30°$

B must be an acute angle as otherwise the sum of the angles would be greater than 180°

$C = 180° - 90° - 30°$
$= 60°$

(ii) $\tan 60° = \sqrt{3}$
$\cos 60° = \frac{1}{2} \Rightarrow \sec 60° = 2$
So $\tan C = \sqrt{3}$ and $\sec C = 2$

(iii) L.H.S. $= \tan^2 C + 1 = \left(\sqrt{3}\right)^2 + 1 = 3 + 1 = 4$
R.H.S. $= \sec^2 C = 2^2 = 4$

So $\tan^2 C + 1 = \sec^2 C$ as required.

You sometimes need to use these identities to solve trigonometric equations.

Example 4
Solve $\operatorname{cosec}^2\theta = 3 + \cot\theta$ for $-180° \le \theta \le 180°$.

The identity $1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$ links $\operatorname{cosec}\theta$ and $\cot\theta$. Substitute this into the equation.

Solution

$\operatorname{cosec}^2\theta = 3 + \cot\theta \Rightarrow 1 + \cot^2\theta = 3 + \cot\theta$
$\Rightarrow \cot^2\theta - \cot\theta - 2 = 0$

This is a quadratic in $\cot\theta$. You can factorise this directly and get $(\cot\theta - 2)(\cot\theta + 1) = 0$, or use the approach shown.

Let $x = \cot\theta$ $\qquad x^2 - x - 2 = 0$

Factorising: $(x-2)(x+1)=0$

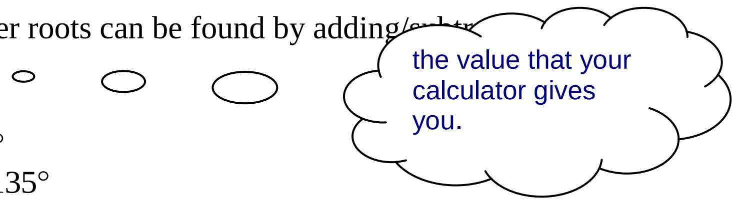
$\Rightarrow x=2$ or $x=-1$

$\Rightarrow \cot\theta=2$ or $\cot\theta=-1$

$$\cot\theta=2\Rightarrow\frac{1}{\tan\theta}=2\Rightarrow\tan\theta=\frac{1}{2}\Rightarrow\theta=26.6°$$

$$\cot\theta=-1\Rightarrow\frac{1}{\tan\theta}=-1\Rightarrow\tan\theta=-1\Rightarrow\theta=-45°$$

Since $y=\tan\theta$ has a period of 180° any other roots can be found by adding/subtr
the principal value.



So the other roots are: $26.6°-180°=-153.4°$

and $-45°+180°=135°$

So the values of θ for which $\text{cosec}^2\theta=3+\cot\theta$ are -153.4°, -45°, 26.6° and 135°.