Section 2: Circular measure

Notes and Examples

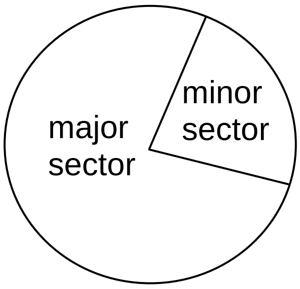
In this section you learn about radians and circular measure.

These notes contain subsections on:

- Sectors of circles
- Small angle approximations

Sectors of circles

- A sector of a circle is the shape enclosed by an arc of the circle and two radii.
  A minor sector is a sector which is smaller than a semi-circle.
  A major sector is a sector which is larger than a semi-circle.



  A sector with angle θ is a fraction of the circle: since the whole circle has angle 2π, the sector is $\frac{\theta}{2\pi}$ of the whole circle.

- The circumference of a circle is $2\pi r$. The length of the arc of a sector with angle θ is $\frac{\theta}{2\pi}$ of the circumference, so the arc length is

  $\frac{\theta}{2\pi} \times 2\pi r = r\theta$.

  Arc length = $r\theta$ where θ is in radians.

- The area of a circle is πr^2. The area of a sector with angle θ is $\frac{\theta}{2\pi}$ of the area, so the area of the sector is $\frac{\theta}{2\pi} \times \pi r^2 = \tfrac{1}{2} r^2 \theta$.:

  Area of a sector = $\frac{1}{2} r^2 \theta$ where θ is in radians.

In this example you need to use the formula for arc length.

Example 1
A sector of a circle with radius 6 cm has an arc length of 2π.
Find the angle subtended at the centre of the circle.

Solution

Use arc length $= r\theta$.

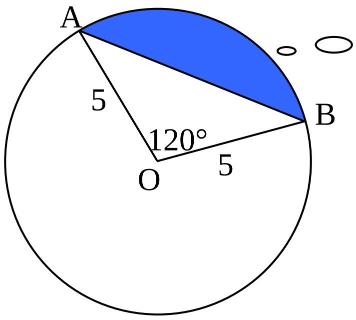
So $2\pi = 6\theta$

So $\theta = \dfrac{2\pi}{6} = \dfrac{\pi}{3}$

In the next example you need to use the formula for sector area.

Example 2

Find the shaded area.



This is called a **segment**.

Solution

Convert 120° to radians: $120° = 120° \times \dfrac{\pi}{180°} = \dfrac{2\pi}{3}$

Area of segment = Area of sector OAB – area of triangle OAB

Area of sector = $\dfrac{1}{2}r^2\theta$ where θ is in radians.

Area of triangle $= \dfrac{1}{2}r^2 \sin\theta$

Area of triangle is ½$ab\sin C$ but a and b are equal to the radius of the circle.

So area of segment $= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta$

$= \tfrac{1}{2}r^2(\theta - \sin\theta)$

Area of segment = $\dfrac{1}{2} \times 5^2\left(\dfrac{2\pi}{3} - \sin\dfrac{2\pi}{3}\right) = 15.4 \text{ cm}^2$

Make sure your calculator is in radians mode.

Small angle approximations

Using radians, the trigonometric functions can be approximated by polynomial functions. If the angle is small, good approximations are given by

$\sin x \approx x$

$\cos x \approx 1 - \tfrac{1}{2}x^2$

$\tan x \approx x$

You can use a spreadsheet or graphing software to see how good these approximations are for small angles.

Example 3

(i) Use small angle approximations to find an expression for

$$\sin 3x + \cos 2x + \tan^2 x \ .$$

(ii) Hence find an approximate value for the smallest positive root of the equation $\sin 3x + \cos 2x + \tan^2 x = 1.5$

Solution

(i)
$$\begin{aligned}\sin 3x + \cos 2x + \tan^2 x &\approx 3x + 1 - \tfrac{1}{2}(2x)^2 + x^2 \\ &= 3x + 1 - 2x^2 + x^2 \\ &= 3x + 1 - x^2\end{aligned}$$

(ii)
$$\sin 3x + \cos 2x + \tan^2 x = 1.5$$
$$3x + 1 - x^2 = 1.5$$
$$2x^2 - 6x + 1 = 0$$
$$x = \frac{6 \pm \sqrt{36 - 4 \times 2 \times 1}}{4} = \frac{6 \pm \sqrt{28}}{4} = \frac{3 \pm \sqrt{7}}{2}$$

The smallest positive root is approximately 0.18.

In the example above, notice that using the small angle approximations give a quadratic equation with two roots. The smaller root, 0.18, is a good approximation to a root of the original equation (try substituting it into the original trig equation – make sure your calculator is in radians!) However, the larger root of the quadratic (2.82) is not such a good approximation, and there are other roots as well (positive and negative) which cannot be approximated by this method. Use graphing software to look at the equation of the curve $y = \sin 3x + \cos 2x + \tan^2 x$, and compare with the quadratic $y = 3x + 1 - x^2$.