Topic assessment

1. (i) By considering turning points, show that $x^3 - 3x^2 + 5 = 0$ has only one real root and that this root lies between –2 and –1. [4]

   (ii) Show that this root is -1.104, correct to 3 d.p. [2]

2. (i) By sketching the line $y = x + 7$ and the curve $y = \frac{1}{8}x^4$, show that the equation $x^4 - 8x - 56 = 0$ has two real roots. [3]

   (ii) Show that the positive root lies between $x = 2$ and $x = 3$. [2]

   (iii) Use the iterative formula $x_{n+1} = \sqrt[4]{8x_n + 56}$, starting from $x = 3$, to find the value of the positive root correct to 2 decimal places. [4]

3. (i) Show that the equation $e^x = x^3 - 1$ has a real root between $x = 2$ and $x = 3$. [2]

   (ii) Use the iterative formula $x_{n+1} = \dfrac{e^{x_n} + 1}{x_n^2}$, starting from $x_0 = 2$, to find two further approximations to the root. [4]

   (iii) Show that the root is 2.081 correct to 3 decimal places. [2]

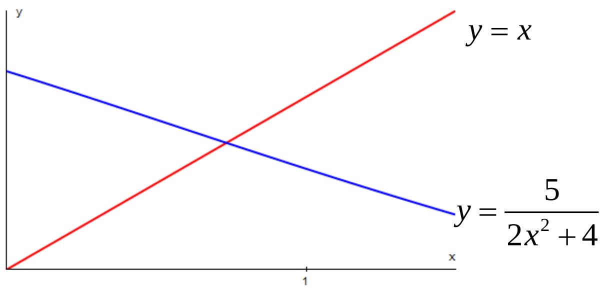
4. (i) Show that the gradient of $y = 2x^3 + 4x - 5$ is always positive and deduce that the equation $2x^3 + 4x - 5 = 0$ has one real root only. [2]

   (ii) Show that this root lies between $x = 0$ and $x = 1$. [2]

   (iii) Show that the equation can be rearranged into the form $x = \dfrac{5}{2x^2 + 4}$. [2]

   (iv) Using the iterative formula $x_{n+1} = \dfrac{5}{2x_n^2 + 4}$ and starting from $x_0 = 1$, find the next two approximations x_1 and x_2 to the root. [4]

   (v) The diagram below shows part of the graphs of $y = x$ and $y = \dfrac{5}{2x^2 + 4}$, and the position of x_0.



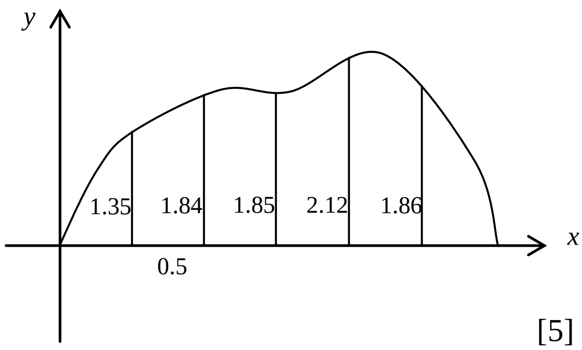
Copy the diagram and draw on it a staircase or cobweb diagram to illustrate how the iterations converge to the root. Indicate the positions of x_1 and x_2 on the x-axis. [2]

(vi) Show that the root is 0.893 correct to 3 decimal places. [2]

5. The root of the equation $x^3 - x + 5 = 0$ is denoted by α.
Taking a first approximation $x_1 = -2$, use the Newton-Raphson method to find the value of α correct to 4 decimal places. [6]

6. The diagram shows a cross-section of a tunnel. The height is measured in metres every 0.5 metres along the cross section. Use the trapezium rule to estimate the area of the cross-section.

Is it an under-estimate or over-estimate?



[5]

7. An estimate is required for the integral $\int_0^1 x\sqrt{x^3+1}\,dx$.

(i) Using 5 rectangles, find overestimates and underestimates for the value of the integral. [6]

(ii) If 20 rectangles were used, find the difference between the overestimate and underestimate for the value of this integral. [3]

(iii) The difference between the overestimate and the underestimate is required to be less than 0.001.
Find the minimum number of rectangles required. [3]

Total 60 marks

Solutions to Topic Assessment

1. (i) $f(x) = x^3 - 3x^2 + 5$

$f'(x) = 3x^2 - 6x$

At turning points, $3x^2 - 6x = 0$

$3x(x-2) = 0$

$x = 0$ or $x = 2$

When x = 0, $f(x) = 5$

When x = 2, $f(x) = 8 - 3\times 4 + 5 = 1$

Both turning points are above the x-axis, so the curve $y = x^3 - 3x^2 + 5$ crosses the x-axis once only.

Therefore the equation $x^3 - 3x^2 + 5 = 0$ has only one real root.

$f(-2) = -8 - 3\times 5 + 5 = -15$

$f(-1) = -1 - 3 + 5 = 1$

There is a change of sign between x = -2 and x = -1, so the root lies between these two values.

[4]

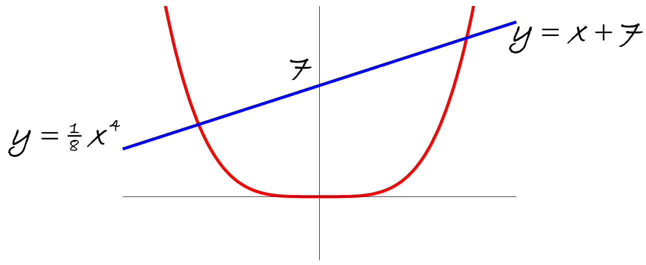
(ii) $f(-1.1035) = (-1.1035)^3 - 3(-1.1035)^2 + 5 = 0.00312$

$f(-1.1045) = (-1.1045)^3 - 3(-1.1045)^2 + 5 = -0.00716$

There is a change of sign between -1.1035 and -1.1045, so the root lies between these two values, and therefore the root is -1.104 to 3 d.p.

[2]

2. (i) At intersections, $\frac{1}{8}x^4 = x + 7 \Rightarrow x^4 = 8(x+7) \Rightarrow x^4 - 8x - 56 = 0$



The graphs intersect at two points, so the equation $x^4 - 8x - 56 = 0$ has two real roots.

[3]

(ii) $f(x) = x^4 - 8x - 56$

$f(2) = 16 - 16 - 56 = -56$

$f(3) = 81 - 24 - 56 = 1$

There is a change of sign between x = 2 and x = 3, so there is a root between these values.

[2]

(iii) $x_{n+1} = \sqrt[4]{8x + 56}$

$x_0 = 3$

$x_1 = \sqrt[4]{8x_0 + 56} = \sqrt[4]{8 \times 3 + 56} = 2.9907$

$x_2 = \sqrt[4]{8x_1 + 56} = \sqrt[4]{8 \times 2.9907 + 56} = 2.9900$

$x_3 = \sqrt[4]{8x_2 + 56} = \sqrt[4]{8 \times 2.9900 + 56} = 2.9899$

The root is 2.99 to 2 decimal places.

[4]

3. (i) $f(x) = e^x - x^3 + 1$

$f(2) = e^2 - 2^3 + 1 = 0.389...$

$f(3) = e^3 - 3^3 + 1 = -5.914...$

So there is a root between x = 2 and x = 3.

[2]

(ii) $x_{n+1} = \dfrac{e^{x_n} + 1}{x_n^2}$

$x_0 = 2$

$x_1 = \dfrac{e^{x_0} + 1}{x_0^2} = \dfrac{e^2 + 1}{2^2} = 2.09726$

$x_2 = \dfrac{e^{x_1} + 1}{x_1^2} = \dfrac{e^{2.09726} + 1}{2.09726^2} = 2.07885$

[4]

(iii) $f(x) = e^x - x^3 + 1$

$f(2.0805) = e^{2.0805} - 2.0805^3 + 1 = 0.00307$

$f(2.0815) = e^{2.0815} - 2.0815^3 + 1 = -0.00191$

There is a change of sign between 2.0805 and 2.0815, so the root lies between these two values, and therefore the root is 2.081 to 3 d.p.

[2]

4. (i) $y = 2x^3 + 4x - 5$

$\dfrac{dy}{dx} = 6x^2 + 4$

The gradient is always positive, so there are no turning points.
Since the value of y is positive for large positive x, and negative for large negative x, the graph must cut the x-axis at least once. Since there are no turning points, it cuts the x-axis once only.
Therefore the equation $2x^3 + 4x - 5 = 0$ has one real root only.

[2]

(ii) $f(x) = 2x^3 + 4x - 5$

$f(0) = -5$

$f(1) = 2 + 4 - 5 = 1$

There is a change of sign between x = 0 and x = 1, so there is a root between these two values.

[2]

(iii) $2x^3 + 4x - 5 = 0$

$2x^3 + 4x = 5$

$x(2x^2 + 4) = 5$

$$x = \frac{5}{2x^2 + 4}$$

[2]

(iv) $x_{n+1} = \dfrac{5}{2x_n^2 + 4}$

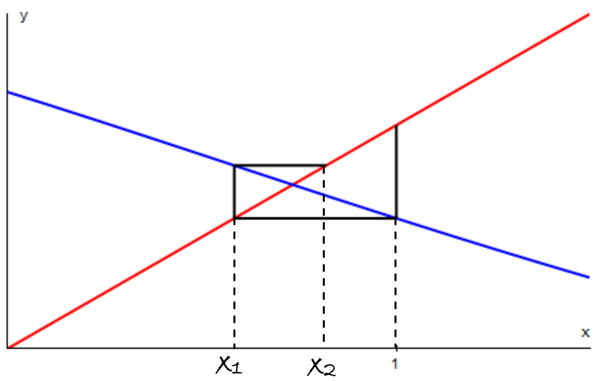
$x_0 = 1$

$$x_1 = \frac{5}{2x_0^2 + 4} = \frac{5}{2 \times 1^2 + 4} = 0.8333$$

$$x_2 = \frac{5}{2x_1^2 + 4} = \frac{5}{2 \times 0.8333^2 + 4} = 0.9278$$

[4]

(v)



[2]

(vi) $f(0.8925) = 2 \times 0.8925^3 + 4 \times 0.8925 - 5 = -0.00815$

$f(0.8935) = 2 \times 0.8935^3 + 4 \times 0.8935 - 5 = 0.000638$

There is a change of sign between 0.8925 and 0.8935, so the root lies between these two values, and therefore the root is 0.893 to 3 d.p.

[2]

5. $f(x) = x^3 - x + 5$

$f'(x) = 3x^2 - 1$

$x_1 = -2$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = -1.90909$$

$x_3 = -1.904175$

$x_4 = -1.904161$

$x_5 = -1.904161$

The root is -1.9042 to 4 d.p.

Check: $f(-1.90415) = 0.0001... > 0$

$f(-1.90425) = -0.0008... < 0$

so the root is -1.9042 to 4 d.p.

6. $h = 0.5$

Using the trapezium rule:

$$\text{Area } = \tfrac{1}{2}h\left[f_0 + f_6 + 2(f_1 + f_2 + f_3 + f_4 + f_5)\right]$$

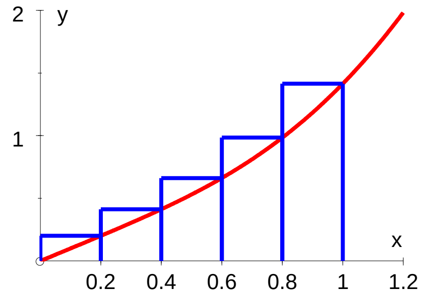
$$= \tfrac{1}{2} \times 0.5\left[0 + 0 + 2(1.35 + 1.84 + 1.85 + 2.12 + 1.86)\right]$$

$$= 4.51$$

This is an underestimate, since most of the trapezia will lie below the actual curve.

[5]

7. (i) Overestimate:

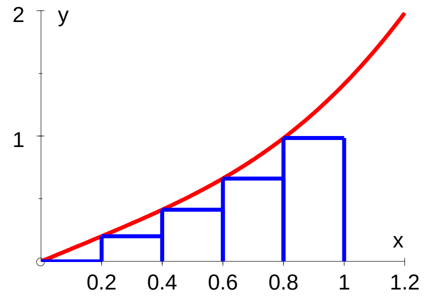


$$\text{Area } = 0.2 \times 0.2\sqrt{0.2^3 + 1} + 0.2 \times 0.4\sqrt{0.4^3 + 1} + 0.2 \times 0.6\sqrt{0.6^3 + 1}$$

$$+ 0.2 \times 0.8\sqrt{0.8^3 + 1} + 0.2 \times 1\sqrt{1^3 + 1}$$

$$= 0.735$$

Underestimate:



$$\text{Area } = 0.2 \times 0\sqrt{0^3 + 1} + 0.2 \times 0.2\sqrt{0.2^3 + 1} + 0.2 \times 0.4\sqrt{0.4^3 + 1}$$

$$+ 0.2 \times 0.6\sqrt{0.6^3 + 1} + 0.2 \times 0.8\sqrt{0.8^3 + 1}$$

$$= 0.452$$

[6]

(ii) Using 20 rectangles, the width of each rectangle would be 0.05

Area of largest rectangle for overestimate $= 0.05 \times 1\sqrt{1^3 + 1}$

Area of smallest rectangle for underestimate $= 0.05 \times 0\sqrt{0^3 + 1}$

Difference between overestimate and underestimate $= 0.05\sqrt{2} - 0$

$= 0.071$

[3]

(iii) Using n rectangles, the width of each rectangle would be $\frac{1}{n}$.

Difference between overestimate and underestimate $= \frac{1}{n}\sqrt{2}$

$$\frac{1}{n}\sqrt{2} < 0.001$$

$$n > 1000\sqrt{2}$$

$$n > 1414.2\ldots$$

At least 1415 rectangles are required.

[3]