Topic assessment

1. Differentiate the following functions.
   (i) $y = x^3 \ln 2x$ [4]
   (ii) $y = \sin^2 x$ [4]
   (iii) $y = x^2 \cos x$ [4]
   (iv) $y = \dfrac{x}{\tan 2x}$ [4]

2. Find an expression in terms of x and y for the gradient of the curve
   $$x^2 + y^2 - 3x + 4y = 6$$
   For what value of y is the tangent to the curve vertical? [5]

3. Find the equation of the tangent to the curve $y = \ln(3x-5)$ at the point where $x = 3$. [5]

4. Show that the curve $y = x - \ln x$ has one turning point only, and give the coordinates of this point. [5]

5. A curve has $y = e^{2x} \cos x$.
   (i) Show that the turning points of the curve occur at the points for which $\tan x = 2$. [5]
   (ii) Find the equation of the normal to the curve at the point for which $x = 0$. [5]

6. For the curve $y = x^2 e^{-x}$,
   (i) Write down the coordinates of the point(s) where the curve cuts the axes. [1]
   (ii) Find the gradient function for the curve and hence the coordinates of any turning points, distinguishing between them. [8]

Total 50 marks

Solutions to topic assessment

1. (i) $y = x^3 \ln 2x$

Let $u = x^3 \quad \Rightarrow \frac{du}{dx} = 3x^2$

Let $v = \ln 2x = \ln 2 + \ln x \Rightarrow \frac{dv}{dx} = \frac{1}{x}$

Using the product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$= x^3 \times \frac{1}{x} + \ln 2x \times 3x^2$$

$$= x^2 + 3x^2 \ln 2x$$

[4]

(ii) $y = \sin^2 x = (\sin x)^2$

Let $u = \sin x \quad \Rightarrow \frac{du}{dx} = \cos x$

$y = u^2 \quad \Rightarrow \frac{dy}{du} = 2u$

Using the chain rule: $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$= 2u \times \cos x$$

$$= 2\sin x \cos x$$

[4]

(iii) $y = x^2 \cos x$

Let $u = x^2 \quad \Rightarrow \frac{du}{dx} = 2x$

Let $v = \cos x \Rightarrow \frac{dv}{dx} = -\sin x$

Using the product rule: $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$$= x^2 \times -\sin x + \cos x \times 2x$$

$$= 2x\cos x - x^2 \sin x$$

[4]

(iv) $y = \frac{x}{\tan 2x}$

Let $u = x \quad \Rightarrow \frac{du}{dx} = 1$

Let $v = \tan 2x \quad \Rightarrow \frac{dv}{dx} = 2\sec^2 2x$

Using the quotient rule: $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

$$= \frac{\tan 2x \times 1 - x \times 2\sec^2 2x}{\tan^2 2x}$$

$$= \frac{\tan 2x - 2x\sec^2 2x}{\tan^2 2x}$$

[4]

2. $x^2 + y^2 - 3x + 4y = 6$

Differentiating implicitly: $2x + 2y\frac{dy}{dx} - 3 + 4\frac{dy}{dx} = 0$

$$(2y+4)\frac{dy}{dx} = 3 - 2x$$

$$\frac{dy}{dx} = \frac{3-2x}{2y+4}$$

The tangent will be vertical where $2y + 4 = 0$

$$y = -2$$

[5]

3. $y = \ln(3x-5)$

$$\frac{dy}{dx} = \frac{1}{3x-5} \times 3 = \frac{3}{3x-5}$$

When x = 3, gradient $= \frac{3}{3\times 3-5} = \frac{3}{4}$

When x = 3, $y = \ln(3\times 3 - 5) = \ln 4 = 2\ln 2$

Equation of tangent is $y - 2\ln 2 = \frac{3}{4}(x-3)$

$$4y - 8\ln 2 = 3x - 9$$

$$4y = 3x + 8\ln 2 - 9$$

[5]

4. $y = x - \ln x$

$$\frac{dy}{dx} = 1 - \frac{1}{x}$$

At turning points, $1 - \frac{1}{x} = 0$

$$\frac{1}{x} = 1$$

$$x = 1$$

The only turning point is at x = 1.

When $x = 1$, $y = 1 - \ln 1 = 1 - 0 = 1$
The turning point is $(1, 1)$.

[5]

5. (i) $y = e^{2x} \cos x$

Let $u = e^{2x} \quad \Rightarrow \frac{du}{dx} = 2e^{2x}$

Let $v = \cos x \quad \Rightarrow \frac{dv}{dx} = -\sin x$

Using the product rule: $\frac{dy}{dx} = -e^{2x} \sin x + 2e^{2x} \cos x$

At turning points, $-e^{2x} \sin x + 2e^{2x} \cos x = 0$

$\sin x = 2\cos x$

$\tan x = 2$

[5]

(ii) $\frac{dy}{dx} = -e^{2x} \sin x + 2e^{2x} \cos x$

When $x = 0$, gradient $= -e^0 \sin 0 + 2e^0 \cos 0 = 2$

Gradient of normal $= -\frac{1}{2}$

When $x = 0$, $y = e^0 \cos 0 = 1$

Equation of normal is $y - 1 = -\frac{1}{2}(x - 0)$

$2(y - 1) = -x$

$2y + x = 2$

[5]

6. (i) $y = x^2 e^{-x}$

When $x = 0$, $y = 0$.

The only point where the curve cuts the axes is $(0, 0)$. [1]

(ii) Let $u = x^2 \quad \Rightarrow \frac{du}{dx} = 2x$

Let $v = e^{-x} \quad \Rightarrow \frac{dv}{dx} = -e^{-x}$

Using the product rule, $\frac{dy}{dx} = -x^2 e^{-x} + 2xe^{-x} = xe^{-x}(2 - x)$

At turning points, $xe^{-x}(2 - x) = 0$

$x = 0$ or $x = 2$

When $x = 0$, $y = 0$

When $x = 2$, $y = \frac{4}{e^2}$

	$x < 0$	$0 < x < 2$	$x > 2$
$\frac{dy}{dx}$	-ve \	+ve /	-ve \

From table, $(0, 0)$ is a minimum point and $\left(2, \frac{4}{e^2}\right)$ is a maximum point.

[8]