Edexcel A level Maths Differential equations

Topic assessment

1. Find y in terms of x given that $\frac{dy}{dx} = x(y-1)$. [4]

2. Solve $(x-1)\frac{dy}{dx} = xy$ for $x > 1$ and $y > 0$, given that $y = 1$ when $x = 3$. [5]

3. Obtain a particular solution to $(1-e^{2y})\frac{dy}{dx} = e^{y}$ given that $y = 0$ when $x = 2$.
   (There is no need to express y in terms of x). [5]

4. Find an expression for y in terms of x given that $x^2\frac{dy}{dx} - y^2 = 0$. [4]

5. At time t seconds the rate of increase in the concentration of flesh eating bugs in a controlled environment is proportional to the concentration C of bugs present. Initially $C = 100$ bugs and after 2 seconds there are five times as many.
   (i) Write down a differential equation connecting $\frac{dC}{dt}$, C and t and hence find an expression for C in terms of t. [7]
   (ii) How many bugs are present after 5 seconds? [2]
   (iii) When will the number of bugs exceed 5000? [3]
   (iv) Find the time at which the concentration of bugs has increased by 50% of the initial concentration. [3]

6. Water is pouring out of a small hole in the bottom of a conical container of height 25 cm. Initially the container is full.
   The rate at which the height x of the water remaining in the container is given by
   $$\frac{dx}{dt} = -\frac{50}{\pi}x^{-\frac{3}{2}}.$$
   (i) Solve the differential equation to find x in terms of t. [5]
   (ii) How long does it take for the container to empty completely? [2]

Total 40 marks

Solutions to topic assessment

1. $\frac{dy}{dx} = x(y-1)$

$$\int \frac{1}{y-1}\,dy = \int x\,dx$$

$$\ln|y-1| = \tfrac{1}{2}x^2 + c$$

$$y - 1 = e^{\frac{1}{2}x^2 + c} = Ae^{\frac{1}{2}x^2}$$

Replacing e^c with A

$$y = 1 + Ae^{\frac{1}{2}x^2}$$

[4]

2. $(x-1)\frac{dy}{dx} = xy$

$$\int \frac{1}{y}\,dy = \int \frac{x}{x-1}\,dx = \int\left(\frac{x-1}{x-1} + \frac{1}{x-1}\right)dx = \int\left(1 + \frac{1}{x-1}\right)dx$$

$$\ln y = x + \ln(x-1) + c$$

$$\ln\left(\frac{y}{x-1}\right) = x + c$$

When $x = 3,\ y = 1 \Rightarrow \ln\left(\frac{1}{2}\right) = 3 + c \Rightarrow c = \ln\left(\frac{1}{2}\right) - 3 = -3 - \ln 2$

$$\ln\left(\frac{y}{x-1}\right) = x - 3 - \ln 2$$

$$\ln\left(\frac{2y}{x-1}\right) = x - 3$$

$$\frac{2y}{x-1} = e^{x-3}$$

$$y = \frac{(x-1)e^{x-3}}{2}$$

[5]

3. $(1 - e^{2y})\frac{dy}{dx} = e^y$

$$\int (1 - e^{2y})e^{-y}\,dy = \int 1\,dx$$

$$\int (e^{-y} - e^{y})\,dy = \int 1\,dx$$

$$-e^{-y} - e^{y} = x + c$$

When $x = 2,\ y = 0 \Rightarrow -1 - 1 = 2 + c \Rightarrow c = -4$

$$-e^{-y} - e^{y} = x - 4$$

$$e^{y} + e^{-y} = 4 - x$$

[5]

4. $x^2\dfrac{dy}{dx} - y^2 = 0$

$x^2\dfrac{dy}{dx} = y^2$

$\displaystyle\int \frac{1}{y^2}\,dy = \int \frac{1}{x^2}\,dx$

$-\dfrac{1}{y} = -\dfrac{1}{x} + c$

$\dfrac{1}{y} = \dfrac{1}{x} - c = \dfrac{1-cx}{x}$

$y = \dfrac{x}{1-cx}$

An equivalent solution is $y = \dfrac{x}{1+kx}$, since $-c$ can be replaced by k.

[4]

5. (i) $\dfrac{dC}{dt} = kC$

$\displaystyle\int \frac{1}{C}\,dC = \int k\,dt$

$\ln C = kt + c$

When $t = 0$, $C = 100 \Rightarrow \ln 100 = c$

$\ln C = kt + \ln 100$

$\ln \dfrac{C}{100} = kt$

$\dfrac{C}{100} = e^{kt}$

$C = 100e^{kt}$

When $t = 2$, $C = 500 \Rightarrow 500 = 100e^{2k} \Rightarrow e^{2k} = 5 \Rightarrow e^k = \sqrt{5}$

$C = 100e^{kt} = 100(e^k)^t = 100(5^{\frac{1}{2}})^t$

$C = 100 \times 5^{\frac{1}{2}t}$

[7]

(ii) When $t = 5$, $C = 100 \times 5^{2.5} = 5590$ (to nearest whole number).

[2]

(iii) $100 \times 5^{\frac{1}{2}t} > 5000$

$5^{\frac{1}{2}t} > 50$

$\frac{1}{2}t \ln 5 > \ln 50$

$t > 4.86$

After 4.86 seconds.

[3]

(iv) $100 \times 5^{\frac{1}{2}t} > 150$

$5^{\frac{1}{2}t} > 1.5$

$\frac{1}{2}t\ln 5 > \ln 1.5$

$t > 0.504$

After 0.504 seconds.

[3]

6. (i) $\int \pi x^{\frac{3}{2}}\,dx = \int -50\,dt$

$\frac{2}{5}\pi x^{\frac{5}{2}} = -50t + c$

When $t = 0, x = 25 \Rightarrow \frac{2}{5}\pi \times 25^{\frac{5}{2}} = c \Rightarrow c = 1250\pi$

$\frac{2}{5}\pi x^{\frac{5}{2}} = 1250\pi - 50t$

$x^{\frac{5}{2}} = 3125 - \frac{125}{\pi}t$

$x = \left(3125 - \frac{125}{\pi}t\right)^{\frac{2}{5}}$

[5]

(iii) When $x = 0$, $\frac{125}{\pi}t = 3125 \quad \Rightarrow t = 25\pi = 78.5$ seconds.

It takes 78.5 seconds (3 s.f.)

[2]