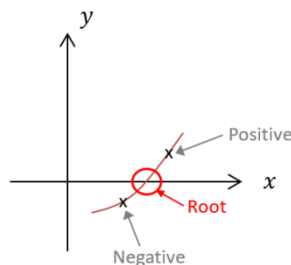


# Summary sheet: Numerical methods

- I1 Locate roots of  $f(x) = 0$  by considering changes of sign of  $f(x)$  in an interval of  $x$  on which  $f(x)$  is sufficiently well-behaved  
Understand how change of sign methods can fail
- I2 Solve equations approximately using simple iterative methods; be able to draw associated cobweb and staircase diagrams  
Solve equations using the Newton-Raphson method and other recurrence relations of the form  $x_{n+1} = g(x_n)$   
Understand how such methods can fail
- I3 Understand and use numerical integration of functions, including the use of the trapezium rule and estimating the approximate area under a curve and limits that it must lie between
- I4 Use numerical methods to solve problems in context

## Locating roots

Remember that a root is when a function = 0 (e.g.  $f(x) = 0$ ), so if you try two values of  $x$  and one answer is positive and the other negative then you must have passed through zero somewhere in between. This means that there is a root between the two values of  $x$  that you tried.



You need to be aware that this method can sometimes fail, e.g.:

- When the curve of  $y = f(x)$  touches the axis (but doesn't go through) – there's a root but no change of sign.
- When the curve of  $y = f(x)$  has a vertical asymptote – there's a change of sign but no root.
- There may be several roots in the interval you are looking at (e.g. the curve crosses the axis several times in between your two values).

## Simple iterative methods

Iterative methods start with an approximate answer and keep refining it so that it gets more and more accurate. To use this method you start by rearranging the equation into the form  $x = g(x)$  for some function  $g$ . You need to have an initial estimate for the answer, then you can use the 'Ans' (answer) key on your calculator. This method does not always converge, or it sometimes gives a different root. If this happens you could try a different rearrangement of the equation.

**To use your calculator 'Ans' key:** Type in your starting value and press = (your starting value is now stored as 'Ans'). Now type in the function  $g(x)$ , substituting every  $x$  with 'Ans', then keep pressing = (don't forget to write each answer down). Every time you press = the calculator puts the new answer into the formula.

# Summary sheet: Numerical methods

e.g. Find the root that lies between 1 and 2 for  $f(x) = x^3 + 2x - 7$

You can quickly check that there is a root between 1 and 2:

$$f(1) = 1^3 + 2(1) - 7 = -4$$

$$f(2) = 2^3 + 2(2) - 7 = 5$$

There is a change of sign and so there must be a root between  $x = 1$  and  $x = 2$ .

The root is when:  $x^3 + 2x - 7 = 0$

Rearranges to give:  $x = \sqrt[3]{7 - 2x}$

Choose a starting value:  $x = 1.5$

On your calculator type:  $1.5=$  (this will store 1.5 as Ans)

Now type:  $\sqrt[3]{7 - 2(\text{Ans})}$  and press = then write down your answer

When you press = again your new answer will be substituted into the equation.

Keep pressing = until the answer stabilises.

## Cobweb and staircase diagrams

Diagrams to illustrate iteration and to determine convergence or divergence. Cobweb diagrams circle around the solution getting closer and closer. Staircase diagrams go up or down to the solution. If the diagram moves outwards instead of inwards then it is divergent instead of convergent.

In general: if  $y = g(x)$

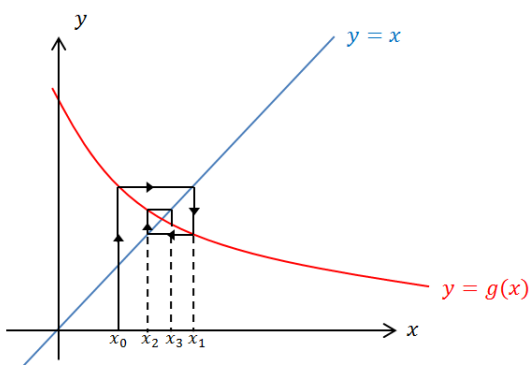
Start by sketching  $y = g(x)$  and the straight line  $y = x$

Now **draw a line** from  $x_0$  (your starting value) **to the curve** ( $y = g(x)$ ) (look at the  $y$  value)

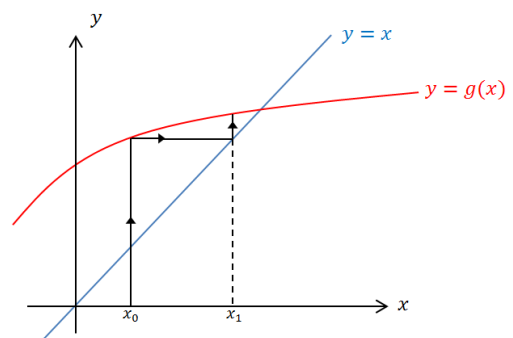
Now (you want to substitute  $x$  with  $y$  so) **draw a line to the line**  $y = x$

Keep repeating this process and see if your diagram converges.

### Cobweb



### Staircase



# Summary sheet: Numerical methods

## Newton-Raphson method

Newton-Raphson is a recurrence relation that uses the previous answer to find a more accurate answer. Every time you press '=' on your calculator the answer should be more accurate. You need to choose a starting value of  $x$  somewhere in the vicinity of the root.

## Newton-Raphson

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

e.g. Show that  $f(x) = 2x^2 - 5x + 1$  has a root between 2 and 3, and use Newton-Raphson to locate this root (to 3 decimal places).

$$f(2) = 2(2)^2 - 5(2) + 1 = -1$$

$$f(3) = 2(3)^2 - 5(3) + 1 = 4$$

There is a change of sign and so there must be a root between  $x = 2$  and  $x = 3$ .

For Newton-Raphson you need to know  $f(x)$  and  $f'(x)$ :

$$f(x) = 2x^2 - 5x + 1$$

$$f'(x) = 4x - 5$$

Let starting value  $(x_0) = 2.5$  so type 2.5 into your calculator and press '='.

$$x_1 = Ans - \frac{2Ans^2 - 5Ans + 1}{4Ans - 5}$$

$$x_1 = 2.3$$

$$x_2 = 2.2809523$$

$$x_3 = 2.280776421$$

$$x_4 = 2.280776406$$

$$x_5 = 2.280776406 = 2.281 \text{ (to 3 d.p.)}$$

Remember that these methods can fail because not all iterations converge to a particular root.

# Summary sheet: Numerical methods

## The trapezium rule

The trapezium rule can be used to estimate the area under a curve when the integration cannot be done directly.

$$A \approx \frac{h}{2} [y_0 + y_n + 2(y_1 + y_2 + \dots + y_{n-1})]$$

Where  $h$  is the width of each strip.

An easy way to do this is to make a table of values (for  $x$  and  $y$ ) and then use the formula – don't be put off if it looks complicated, just remember:

$$A \approx \frac{h}{2} (\textit{First} + \textit{last} + 2 \times (\textit{everything in the middle}))$$

e.g. Approximate the following interval using the trapezium rule with 4 strips:  $\int_1^3 e^{x^2-2x} dx$

First find  $h$ :  $h = \frac{3-1}{4} = 0.5$

Next do a table, going up in jumps of 0.5:

$x$	1	1.5	2	2.5	3
$y (e^{x^2-2x})$	0.36788	0.47237	1	3.49034	20.08554

n.b. It is correct to have 5 values for 4 strips as you include each end of the interval.

Now use:  $A \approx \frac{h}{2} (\textit{First} + \textit{last} + 2 \times (\textit{everything in the middle}))$

Substituting your values in:  $A \approx \frac{0.5}{2} (0.36788 + 20.08554 + 2 \times (0.47237 + 1 + 3.49034))$   
 $= 7.59471$