Summary sheet: Parametric equations

C3 Understand and use the parametric equations of curves and conversion between Cartesian and parametric forms
C4 Use parametric equations in modelling in a variety of contexts
G5 (part) Differentiate simple functions and relations defined parametrically, for first derivative only

Parametric equations of curves

You will be familiar with sketching curves using Cartesian equations, (i.e. using x and y coordinates). Parametric equations are just a different way of expressing them.

Cartesian equations: Gives the direct relationship between x and y. **e.g.** $\boldsymbol{y = x^2 - 7x + 3}$

Parametric equations: Uses a 3rd variable (usually t or θ) to define x and y. **e.g.** $\boldsymbol{x = t + 3}$, $\boldsymbol{y = 2t^2}$

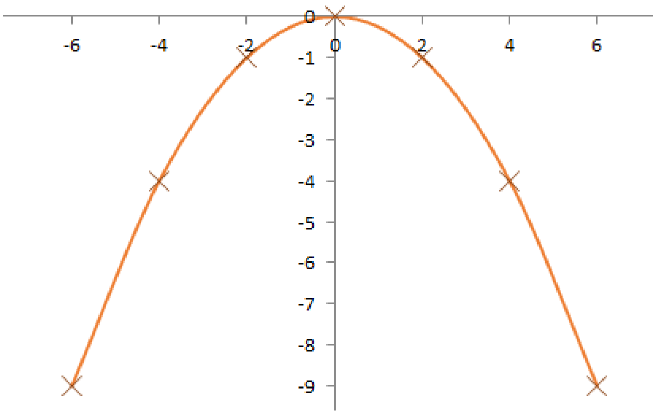
Sketching parametric equations of curves

Don't let the extra variable put you off. Just find some values of x and y (coordinates), plot them and join them as a curve.

e.g. Sketch the curve defined by $\boldsymbol{x = 2t}$ and $\boldsymbol{y = -t^2}$ between $\boldsymbol{t = -3}$ *and* $\boldsymbol{3}$.
It's probably easiest to use a table to find the values of x and y:

t	-3	-2	-1	0	1	2	3
$x\ (= 2t)$	-6	-4	-2	0	2	4	6
$y\ (= -t^2)$	-9	-4	-1	0	-1	-4	-9

Plot your points (x, y) and join them with a curve:



Converting between parametric and cartesian

The general idea is to combine the equations to eliminate t. You can do whatever seems easier, e.g. rearrange one of the equations to make t the subject and then substitute into the other equation OR rearrange them both to make t the subject and then equate them.

e.g. Convert $\boldsymbol{x = t + 3}$ and $\boldsymbol{y = 2t^2}$ into Cartesian form.

$\boldsymbol{x = t + 3}$ $\rightarrow$ Rearranges to: $t = x - 3$
$\boldsymbol{y = 2t^2}$ $\rightarrow$ Substitute t to give: $y = 2(x-3)^2$
$\therefore\ y = 2x^2 - 12x + 18$

OR: $t = x - 3$ and $t = \sqrt{\frac{y}{2}}$

$\therefore x - 3 = \sqrt{\frac{y}{2}} \rightarrow \frac{y}{2} = (x-3)^2$

$\therefore y = 2(x-3)^2$

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If your parametric equations contain trig functions: 1st find a trigonometric identity that connects them, then rearrange the parametric equations so that you can substitute them into the identity.

e.g. Convert $x = \cos 2\theta - 3$ and $y = \frac{\sin\theta}{2}$ into cartesian form.

1st look for a trig identity that contains both $\cos 2\theta$ and $\sin\theta$: $\mathbf{\cos 2\theta = 1 - 2\sin^2\theta}$

Rearrange the equations: $\cos 2\theta = x + 3$
$\sin\theta = 2y$

Identity to use: $\cos 2\theta = 1 - 2\sin^2\theta$
Sub into identity: $x + 3 = 1 - 2(2y)^2$

Tidy up: $8y^2 = 1 - x - 3$

$$y = \sqrt{\frac{-2-x}{8}}$$

Parametric equations of Circles

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$ Gives a circle with radius r and centre $(0,0)$

$$\left.\begin{aligned} x &= a + r\cos\theta \\ y &= b + r\sin\theta \end{aligned}\right\}$$ Gives a circle with radius r and centre (a,b)

To convert to Cartesian you can use the methods shown above OR just remember the general equation of a circle ($(x-a)^2 + (y-b)^2 = r^2$) and substitute in your values of a, b and r.

Differentiating parametric equations

If $\quad x = \mathrm{f}(t)$ and $y = \mathrm{g}(t)$

Then $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \div \frac{\mathrm{d}x}{\mathrm{d}t}$$

Think about why this works and it will help you to remember it:

$$\frac{dy}{dt} \div \frac{dx}{dt} = \frac{dy}{\cancel{dt}} \times \frac{\cancel{dt}}{dx} = \frac{dy}{dx}$$

e.g. find $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 2t + 3$ and $y = 3t^2$

$\frac{\mathrm{d}y}{\mathrm{d}t} = 6t \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = 2 \qquad \therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6t}{2} = 3t$

A good way to remember which way round to do them is that you want $\mathrm{d}y$ at the top so that needs to come 1st then divide by $\mathrm{d}x$ as you want it at the bottom.

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Integrating with parametric equations

To integrate using parametric equations, you can use the chain rule so that the variable involved is the parameter.

If	$\boldsymbol{x} = \mathbf{f}(\boldsymbol{t})$ **and** $\boldsymbol{y} = \mathbf{g}(\boldsymbol{t})$
Then	$\int \boldsymbol{y}\,\mathbf{d}\boldsymbol{x} = \int \boldsymbol{y}\frac{\mathrm{d}x}{dt}\mathbf{d}\boldsymbol{t}$

If you are using definite integration (e.g. to find the area under a parametric curve) then the limits must be values of t, rather than x.