Summary sheet: Integration

H2 Integrate e^{kx}, 1/x, sin kx, cos kx and related sums, differences and constant multiples
H3 Evaluate definite integrals; use a definite integral to find the area between two curves
H4 Understand and use integration as the limit of a sum
H5 Carry out simple cases of integration by substitution and integration by parts; understand these methods as the inverse processes of the chain and product rules respectively (Integration by substitution includes finding a suitable substitution and is limited to cases where one substitution will lead to a function which can be integrated; integration by parts includes more than one application of the method but excludes reduction formulae)
H6 Integrate using partial fractions that are linear in the denominator

Some integrals you need to know:

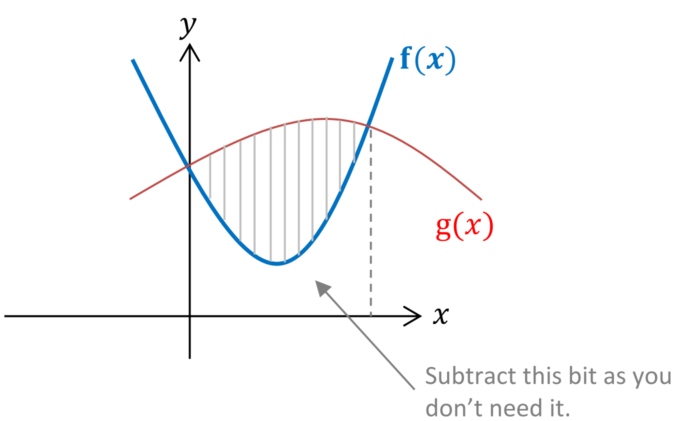
$\int \frac{1}{x}\,\mathrm{d}x = \ln x + c$	e.g. $\int \frac{2}{x}\,\mathrm{d}x = 2\ln x + c$
The following use the **reverse chain rule** (i.e. the chain rule says X by the derivative of the 'inside function' so the reverse chain rule says ÷ by the derivative of the 'inside function').	
$\int e^{kx}\,\mathrm{d}x = \frac{e^{kx}}{k} + c$	e.g. $\int e^{5x}\,\mathrm{d}x = \frac{e^{5x}}{5} + c$
$\int \sin kx\,\mathrm{d}x = -\frac{\cos kx}{k} + c$	e.g. $\int \sin 3x\,\mathrm{d}x = -\frac{\cos 3x}{3} + c$
$\int \cos kx\,\mathrm{d}x = \frac{\sin kx}{k} + c$	e.g. $\int \cos 7x\,\mathrm{d}x = \frac{\sin 7x}{7} + c$

Finding the area between 2 curves

See the AS Summary sheet (integration) for a reminder of how to do definite integration and find the area under a curve. The same method can be used to find the area between 2 curves and you just have to remember to subtract the area you don't need.

The best way is to do a quick sketch so that you know where the curves are, and then you can subtract the correct area. You will also need to find out where they intersect.

e.g. Find the area between $f(x)$ and $g(x)$:



Find where the curves intersect by using simultaneous equations.

You can see from the sketch that to find the area between the curves you need to do: area below $g(x)$ – area below $f(x)$.

The easiest way is to find the integral of $(g(x) - f(x))$ between the x-values of the points of intersection.

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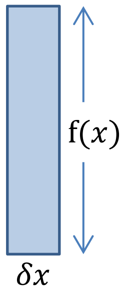
Integration as the limit of a sum

Definite integration calculates the area under a curve by splitting the area into lots of little rectangles, finding the area of each rectangle, and then adding all the areas. The height of each rectangle is y (or $\mathrm{f}(x)$) and the width of each rectangle will be δx (small change in x). As this small change gets smaller (i.e. lots of very thin rectangles) the answer gets more accurate. i.e. as $\delta x \to 0$ the answer is the most accurate it can be. The area of each rectangle is $\boxed{\mathrm{f}(x) \times \delta x}$ (height X width) and so:

$$\boxed{\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \lim_{\delta x \to 0} \sum_a^b \mathrm{f}(x)\delta x}$$



Integration by substitution

Remember that when you differentiate using the chain rule you substitute your function of x with u. You can also use this method of substitution with integration by choosing a function ($u(x)$) (which can be differentiated) to help you to integrate a tricky function. You need to substitute all $x's$ (including $\mathrm{d}x$) with terms involving u.

e.g. find $\displaystyle\int (\mathbf{4x-3})(\mathbf{3x+1})^3 \ \mathbf{dx}$

The $(3x+1)^3$ is the tricky part so define a new variable: $u = 3x+1$ (then use this to find x and $\mathrm{d}x$)

$u = 3x + 1$

Differentiates to give: $\frac{du}{dx} = 3 \quad \to \quad \therefore \mathrm{d}x = \frac{\mathrm{d}u}{3}$

$u = 3x + 1$

Rearranges to give: $x = \frac{u-1}{3}$

Now substitute all u terms into the original problem:

$$\int (4x-3)(3x+1)^3\ \mathrm{d}x = \int \left(4\left(\frac{u-1}{3}\right) - 3\right)(u)^3\,\frac{\mathrm{d}u}{3}$$

$$= \int \left(\frac{4}{3}u - \frac{4}{3} - 3\right)\frac{u^3}{3}\,\mathrm{d}u$$

$$= \int \left(\frac{4}{9}u^4 - \frac{13}{9}u^3\right)\,\mathrm{d}u$$

$$= \frac{4}{45}u^5 - \frac{13}{36}u^4 + c$$

To finish the integral off substitute every u with $3x+1$

$$= \frac{4}{45}(3x+1)^5 - \frac{13}{36}(3x+1)^4 + c$$

N.b. if you were doing **definite integration** you would also need to **change the limits**.
For the example above, if the limits were $x = 2$ and $x = 5$ you would need to replace them:

	When $x = 2$	$u = 3x + 1 = 3(2) + 1 = 7$
And	when $x = 5$	$u = 3x + 1 = 3(5) + 1 = 16$

Integration by parts

This is used to integrate a function multiplied by a function and so it is the equivalent of the product rule for differentiation BUT instead of calling the functions u and v you call the functions u and $\frac{dv}{dx}$. This means that you will differentiate one term and integrate the other.

When choosing which function to call u and which to call $\frac{dv}{dx}$ there is an order of preference for u that will make the integration easier:

$$\boxed{\begin{array}{l} \ln(x) \\ x \\ e^x \end{array}}$$

Decide which is which then write down u, v, $\frac{du}{dx}$ and $\frac{dv}{dx}$ and substitute them into the formula.

$$\boxed{\int u\frac{\mathbf{d}v}{\mathbf{d}x}\mathbf{d}x = uv - \int v\frac{\mathbf{d}u}{\mathbf{d}x}\mathbf{d}x}$$

Don't forget you have to integrate the second part of the formula.

e.g. $\displaystyle\int x^2 \ln x \, dx$

Using order of preferences choose $u = lnx$

First find the 4 things you need:

differentiate	$u = \ln x$	$v = \frac{x^3}{3}$	integrate
	$\frac{du}{dx} = \frac{1}{x}$	$\frac{dv}{dx} = x^2$	

Then substitute in the formula: $\displaystyle\int x^2 \ln x \, dx = \ln x \frac{x^3}{3} - \int \frac{x^3}{3} \times \frac{1}{x} dx$

Tidy up: $\displaystyle= \ln x \frac{x^3}{3} - \int \frac{x^2}{3} dx$

Integrate final part: $\displaystyle= \ln x \frac{x^3}{3} - \frac{x^3}{9} + c$

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N.B. If you still have 2 functions multiplied together inside the new integral you will have to use integration by parts again. It is a long process so be careful, and systematic, and remember that the formula says subtract ALL of the new integral so you will need to keep this in a bracket.

e.g. $\displaystyle\int \boldsymbol{x^2\,e^x\,\mathrm{d}x}$

First find the 4 things you need:

differentiate			integrate
	$\boldsymbol{u = x^2}$	$v = e^x$	
	$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$	$\boldsymbol{\dfrac{\mathrm{d}v}{\mathrm{d}x} = e^x}$	

You need to apply integration by parts again

Then substitute in the formula: $\displaystyle\int x^2 e^x\,\mathrm{d}x = x^2 e^x - \int e^x 2x\,\mathrm{d}x$

differentiate			integrate
	$\boldsymbol{u = 2x}$	$v = e^x$	
	$\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2$	$\boldsymbol{\dfrac{\mathrm{d}v}{\mathrm{d}x} = e^x}$	

Substitute again: $\displaystyle = x^2 e^x - \left(2xe^x - \int 2e^x\,\mathrm{d}x\right)$

Integrate final part & remove bracket: $= x^2 e^x - 2xe^x + 2e^x + c$

You could factorise to give a final answer of: $e^x(x^2 - 2x + 2) + c$

Integration using partial fractions

e.g. $\displaystyle\int \boldsymbol{\frac{x+7}{(x+1)(x+3)}\,\mathrm{d}x}$

An integration like this looks tricky. However, you can use partial fractions to split it into 2 separate fractions and make it easier to integrate. See Algebra summary sheet for more information on partial fractions.

$$\int \frac{x+7}{(x+1)(x+3)}\,\mathrm{d}x \;=\; \int \frac{3}{x+1} - \frac{2}{x+3}\,\mathrm{d}x \;=\; 3\ln(x+1) - 2\ln(x+3) + c$$