Summary sheet: Further differentiation

G1 (part) differentiation from first principles for sin x and cos x
G2 (part) Differentiation e^{kx} and a^{kx}, sin kx, cos kx, tan kx and related sums, differences and constant multiples. Understand and use the derivative of ln x
G5 (part) Differentiate simple functions and relations defined implicitly, for first derivative only

Differentiate $\sin x$ and $\cos x$ from first principles

You will need to remember:

Compound angle formulae:

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$
$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

Small angle approximations of sine and cosine:

$$\sin\theta \approx \theta \qquad \cos\theta \approx 1 - \frac{\theta^2}{2}$$

$$\therefore \frac{\sin\theta}{\theta} \approx 1 \qquad \frac{\cos\theta - 1}{\theta} \approx -\frac{\theta}{2}$$

$\sin x$:

$$y = \sin x$$

Add the small changes
$$y + \delta y = \sin(x + \delta x) \quad (1)$$

Original
$$y = \sin x \quad (2)$$

Subtract (1-2) to leave δy on its own
$$\delta y = \sin(x + \delta x) - \sin x$$

Use trig identity (with $A = x$ and $B = \delta x$)
$$\delta y = \sin x \cos\delta x + \cos x \sin\delta x - \sin x$$

Divide by δx to get $\frac{\delta y}{\delta x}$
$$\frac{\delta y}{\delta x} = \frac{\sin x \cos\delta x + \cos x \sin\delta x - \sin x}{\delta x}$$

Factorise numerator:
$$\frac{\delta y}{\delta x} = \frac{\sin x(\cos\delta x - 1) + \cos x \sin\delta x}{\delta x}$$

Write as separate fractions:
$$\frac{\delta y}{\delta x} = \frac{\sin x(\cos\delta x - 1)}{\delta x} + \frac{\cos x \sin\delta x}{\delta x}$$

Substitute small angle approx.:
$$\frac{\delta y}{\delta x} = \sin x\left(-\frac{\delta x}{2}\right) + \cos x\,(1)$$

So as $\delta x \to 0$ you have:
$$\frac{\delta y}{\delta x} \to \frac{\mathrm{d}y}{\mathrm{d}x} = \sin x\,(0) + \cos x$$

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$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos x$$

$\cos x$:

$$y = \cos x$$

Add the small changes $\qquad y + \delta y = \cos(x + \delta x) \qquad (1)$

Original $\qquad y = \cos x \qquad (2)$

Subtract (1-2) to leave δy on its own $\qquad \delta y = \cos(x + \delta x) - \cos x$

Use trig identity
(with $A = x$ and $B = \delta x$)

$$\delta y = \cos x \cos \delta x - \sin x \sin \delta x - \cos x$$

Divide by δx to get $\frac{\delta y}{\delta x}$

$$\frac{\delta y}{\delta x} = \frac{\cos x \cos \delta x - \sin x \sin \delta x - \cos x}{\delta x}$$

Factorise numerator:

$$\frac{\delta y}{\delta x} = \frac{\cos x\,(\cos \delta x - 1) - \sin x \sin \delta x}{\delta x}$$

Write as separate fractions:

$$\frac{\delta y}{\delta x} = \frac{\cos x\,(\cos \delta x - 1)}{\delta x} - \frac{\sin x \sin \delta x}{\delta x}$$

Substitute small angle approx.:

$$\frac{\delta y}{\delta x} = \cos x \left(-\frac{\delta x}{2}\right) - \sin x\,(1)$$

So as $\delta x \to 0$ you have:

$$\frac{\delta y}{\delta x} \to \frac{\mathrm{d}y}{\mathrm{d}x} = \cos x\,(0) - \sin x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\sin x$$

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Differentiating exponential, log and trig functions:

Remember:

$y =$	$\frac{\mathbf{d}y}{\mathbf{d}x} =$	
e^x	e^x	
a^x	$a^x \ln a$	*
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\tan x$	$\sec^2 x$	
$\ln x$	$\frac{1}{x}$	

* It is a good idea to make sure you understand how to differentiate a^x by taking ln of each side then differentiating (using implicit differentiation on the left hand side) to get the result.

Using the chain rule

If you have a function of a function (e.g. e^{3x}) you will have to use the chain rule (see previous summary sheet "differentiation" for further explanation). You can think of the chain rule as: the derivative of the outside function x the derivative of the inside function. So, if you have to differentiate any of the above using the chain rule you will need to differentiate the inside function as well as the outside. The following table is a summary of the chain rule for the above functions:

$y =$	$\frac{\mathbf{d}y}{\mathbf{d}x} =$	Example	Process Outside x inside	Answer
e^{kx}	$k\mathrm{e}^x$	e^{7x}	$\mathrm{e}^{7x} \times 7$	$7\mathrm{e}^{7x}$
a^{kx}	$ka^{kx} \ln a$	5^{8x}	$5^{8x} \ln 5 \times 8$	$8 \ln (5) 5^{8x}$
$\sin kx$	$k\cos kx$	$\sin 5x$	$\cos 5x \times 5$	$5 \cos 5x$
$\cos kx$	$-k\sin kx$	$\cos 2x$	$-\sin 2x \times 2$	$-2 \sin 2x$
$\tan kx$	$k\sec^2 kx$	$\tan 3x$	$\sec^2 3x \times 3$	$3 \sec^2 3x$
$\ln kx$	$\frac{1}{x}$	$\ln 4x$	$\frac{1}{4x} \times 4$	$\frac{1}{x}$

If a function has a few terms just differentiate each term separately.

e.g. differentiate $y = 3\mathrm{e}^{4x} + 2 \sin x - \ln x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 12\mathrm{e}^{4x} + 2 \cos x - \frac{1}{x}$$

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Implicit differentiation

You can use implicit differentiation when you want to find $\frac{\mathrm{d}y}{\mathrm{d}x}$ but the $x's$ and $y's$ are mixed up and it is tricky to rearrange/separate them.

The method uses the chain rule but an easy way to remember what to do is:
Differentiate every term separately and when you have to differentiate with respect to y instead of x just multiply that term by $\frac{\mathrm{d}y}{\mathrm{d}x}$ to adjust it. Then rearrange to find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

e.g. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the function $x^2 + 3y = 2x$

$3y$ had to be differentiated with respect to y so multiply by $\frac{\mathrm{d}y}{\mathrm{d}x}$

$$2x + 3\frac{\mathrm{d}y}{\mathrm{d}x} = 2$$

$$3\frac{\mathrm{d}y}{\mathrm{d}x} = 2 - 2x$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2 - 2x}{3}$$

The product, quotient and chain rule

Sometimes you will have to use one of the 3 rules as well as implicit differentiation.

e.g. Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ for the function $(x - y)^2 = 5x$

$$u = -2x \qquad v = y$$
$$\frac{\mathrm{d}u}{\mathrm{d}x} = -2 \qquad \frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}x}$$

Multiply out the brackets: $x^2 - 2xy + y^2 = 5x$

Differentiate every term, remembering to use the product rule for $-2xy$: $2x - 2x\frac{\mathrm{d}y}{\mathrm{d}x} - 2y + 2y\frac{\mathrm{d}y}{\mathrm{d}x} = 5$

Factorise $\frac{\mathrm{d}y}{\mathrm{d}x}$: $\frac{\mathrm{d}y}{\mathrm{d}x}(-2x + 2y) = 5 - 2x + 2y$

Tidy up: $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{5 - 2x + 2y}{-2x + 2y}$