Summary sheet: Trigonometric identities

E6 Understand and use double angle formulae, use of formulae for sin(A±B), cos(A±B) and tan(A±B); understand geometric proofs of these formulae. Understand and use expressions for a cos θ + b sin θ in the equivalent forms of r cos (θ±α) or r sin(θ ±α)
E8 Construct proofs involving trigonometric functions and identities
E9 Use trigonometric functions to solve problems in context, including problems involving vectors, kinematics and forces

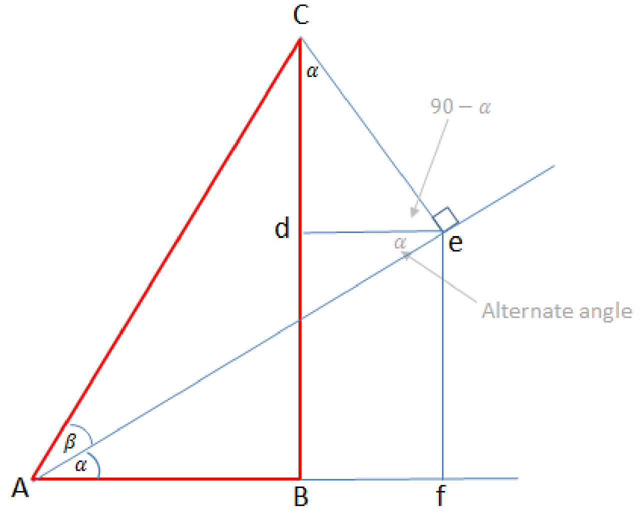
Compound angle formulae

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$$

Proof of formulae



From the red triangle:

$$\sin(\alpha + \beta) = \frac{BC}{AC}$$

From the diagram:

$$BC = Cd + ef$$

$$\therefore \sin(\alpha + \beta) = \frac{ef + Cd}{AC} = \frac{ef}{AC} + \frac{Cd}{AC}$$

$$\therefore \sin(\alpha + \beta) = \frac{ef}{AC} \times \frac{Ae}{Ae} + \frac{Cd}{AC} \times \frac{Ce}{Ce}$$

Remember you can X top & bottom by the same thing without changing the fraction.

$$\therefore \sin(\alpha + \beta) = \frac{ef}{Ae} \times \frac{Ae}{AC} + \frac{Cd}{Ce} \times \frac{Ce}{AC}$$

∴ Using the smaller triangles:

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

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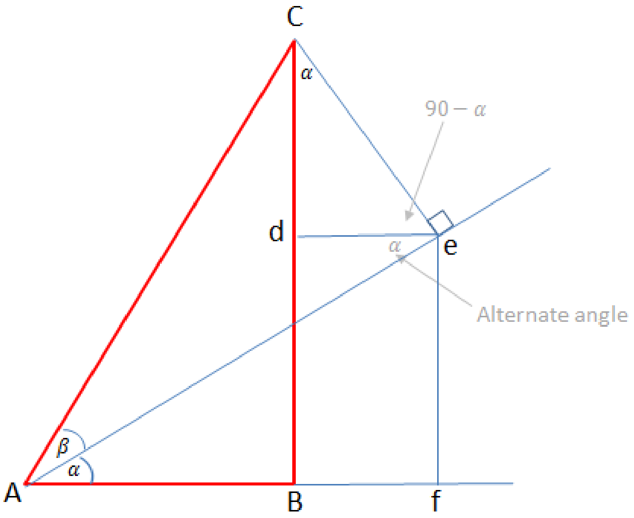

From the red triangle:

$$\boldsymbol{\cos(\alpha+\beta) = \frac{AB}{AC}}$$

From the diagram:

$$AB = Af - de$$

$$\therefore \cos(\alpha+\beta) = \frac{Af-de}{AC} = \frac{Af}{AC} - \frac{de}{AC}$$

$$\therefore \cos(\alpha+\beta) = \frac{Af}{AC} \times \frac{Ae}{Ae} - \frac{de}{AC} \times \frac{Ce}{Ce}$$

$$\therefore \cos(\alpha+\beta) = \frac{Af}{Ae} \times \frac{Ae}{AC} - \frac{de}{Ce} \times \frac{Ce}{AC}$$

∴ Using the smaller triangles:

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$$

To prove the difference formulae just replace $+\beta$ with $-\beta$ and use the identities:

$$\cos(-\beta) = \cos(\beta)$$

$$\sin(-\beta) = -\sin(\beta)$$

Finally remember that: $\tan(\alpha+\beta) = \frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$

$$\therefore \tan(\alpha+\beta) = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}$$

$$\therefore \tan(\alpha+\beta) = \frac{\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

Divide top and bottom by $\cos\alpha\cos\beta$

$$= \frac{\frac{\sin\alpha\cancel{\cos\beta}}{\cos\alpha\cancel{\cos\beta}} + \frac{\cancel{\cos\alpha}\sin\beta}{\cancel{\cos\alpha}\cos\beta}}{\frac{\cancel{\cos\alpha}\cancel{\cos\beta}}{\cancel{\cos\alpha}\cancel{\cos\beta}} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\therefore \tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

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Double angle formulae

The double angle formulae can be derived from the compound angle formulae. Just let $A = B$ in each case and you will find the double angle formulae.

$\sin(A + A) = \sin A\cos A + \cos A\sin A = 2\sin A\cos A$
$\cos(A + A) = \cos A\cos A - \sin A\sin A = \cos^2 A - \sin^2 A$
$\tan(A + A) = \dfrac{\tan A + \tan A}{1 - \tan A\tan A} = \dfrac{2\tan A}{1 - \tan^2 A}$

You can find 2 further identities for $\cos 2A$ by using the identity $\cos^2 A + \sin^2 A = 1$ as follows:

So far you know that: $\cos 2A = \cos^2 A - \sin^2 A$
Replace $\cos^2 A$ with $1 - \sin^2 A$: $\cos 2A = 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A$
OR Replace $\sin^2 A$ with $1 - \cos^2 A$: $\cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1$

Summary:

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A \text{ OR } 1 - 2\sin^2 A \text{ OR } 2\cos^2 A - 1$$

$$\tan 2A = \frac{2\tan A}{1 - \tan^2 A}$$

The forms $r\cos(\theta \pm \alpha)$ and $r\sin(\theta \pm \alpha)$

$$a\sin\theta \pm b\cos\theta = r\sin(\theta \pm \alpha)$$

$$a\cos\theta \pm b\sin\theta = r\cos(\theta \mp \alpha)$$

where $r = \sqrt{a^2 + b^2}$, $\cos\alpha = \frac{a}{r}$ and $\sin\alpha = \frac{b}{r}$

it follows that $\tan\alpha = \frac{b}{a}$

E.g. Write $5\sin\theta + 2\cos\theta$ in the form $r\sin(\theta \pm \alpha)$

$r = \sqrt{a^2 + b^2} = \sqrt{5^2 + 2^2} = \sqrt{29}$

$\tan\alpha = \frac{b}{a}$, $\alpha = \arctan\left(\frac{2}{5}\right) = 21.8°$

$\therefore$ $\mathbf{5\sin\theta + 2\cos\theta = \sqrt{29}\sin(\theta + 21.8°)}$