Summary sheet: Algebra

D1 Understand and use the binomial expansion to any rational n, including its use for approximation; be aware that the expansion is valid for $\left|\frac{bx}{a}\right| < 1$ (proof not required)
B6 (part) Simplify rational expressions including by factorising and cancelling, and algebraic division (by linear expressions only)
B10 Decompose rational functions into partial fractions (denominators not more complicated than squared linear terms and with no more than 3 terms, numerators constant or linear)

See AS summary sheets for previous work.

The binomial expansion

You have already seen how to use the binomial expansion for $(a+b)^n$ where n is a positive integer (see AS summary sheet). In this case there are a finite number of terms.

However, when using the binomial expansion where n is NOT a positive integer, there will be an infinite number of terms. This means that the **binomial expansion can only be used when $-1 < x < 1$** because then x^n will decrease as n increases and there will be a limit.

Remember:

$$\boxed{\boldsymbol{(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 \ldots\ldots}}$$

If you take note of the pattern it will make this easier to remember (multiply **2** n terms and use x to the power of **2** on the top, and **2**! on the bottom, multiply **3** n terms and use x to the power of **3** on the top, and **3**! on the bottom etc).

e.g. Expand $(1+x)^{-4}$ up to the x^3 term

3 brackets

Power of 3

$n = -4$

$$(1+x)^{-4} = 1 + (-4)x + \frac{(-4)(-5)x^2}{2!} + \frac{(-4)(-5)(-6)x^3}{3!} \ldots\ldots$$

3!

$$= 1 - 4x + 10x^2 - 20x^3$$

Valid when $-1 < x < 1$

Different 2nd term

If the 2nd term in the bracket is different you can still expand as above but you need to be careful:

- Remember to do the whole 2nd term to the power
- Be careful if there is a negative number; always do the whole term to the power.

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e.g. Expand $\sqrt[4]{1-3x}$ up to the x^2 term

First rewrite the question as $(1-3x)^{\frac{1}{4}}$ (and remember that you will be using $-3x$ instead of x).

$$(1-3x)^{\frac{1}{4}} = 1 + \tfrac{1}{4}(-3x) + \frac{\left(\frac{1}{4}\right)\left(\frac{1}{4}-1\right)(-3x)^2}{2!}$$

$$= 1 - \tfrac{3}{4}x - \frac{27x^2}{32}$$

$n = \frac{1}{4}$

Replace x with $-3x$

Valid when $-1 < 3x < 1$ $\quad\therefore\quad$ $-\frac{1}{3} < x < \frac{1}{3}$ $\quad$ can be written as $|x| < \frac{1}{3}$

Different 1st term

If the 1st term in the bracket is not 1, you can still use the expansion. You will need to take out a factor to make the 1st term=1.

Remember: $\boxed{(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n}$

e.g. Expand $(3-x)^{-3}$ up to the x^3 term

First rewrite the question as: $3^{-3}\left(1-\frac{x}{3}\right)^{-3}$

$n = -3$

Replace x with $-\frac{x}{3}$

$$3^{-3} = \frac{1}{27}$$

$$\left(1-\frac{x}{3}\right)^{-3} = 1 + (-3)\left(-\frac{x}{3}\right) + \frac{(-3)(-4)\left(-\frac{x}{3}\right)^2}{2!} + \frac{(-3)(-4)(-5)\left(-\frac{x}{3}\right)^3}{3!}$$

$$3^{-3}\left(1-\frac{x}{3}\right)^{-3} = \frac{1}{27}\left(1 + x + \frac{2x^2}{3} + \frac{10x^3}{27}\right)$$

Valid when $-1 < \frac{1}{3}x < 1$ $\quad\therefore\quad$ $-3 < x < 3$ $\quad$ can be written as $|x| < 3$

Simplifying rational expressions

Factorising

If you can take a common factor out of the numerator and the denominator then you can simplify the expression by cancelling. Always make sure you **factorise first** so that you are not tempted to cancel something that does not appear in every term.

e.g. Simplify $\frac{5x^2y^3 - 10x^3y^2}{10x^2y^5}$

Factorise top & bottom: $\frac{\cancel{5x^2y^2}(y-2x)}{\cancel{5x^2y^2}(2y^3)}$

Cancels (simplifies) to: $\frac{(y-2x)}{(2y^3)}$

See AS summary sheet (Polynomials) for more advice.

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Algebraic division

See AS summary sheet (Polynomials)

Decomposing rational functions into partial fractions

Sometimes it can be useful to split an algebraic fraction into 2, or more, separate (partial) fractions.

For example: $$\frac{7x-13}{(x-3)(x+1)} = \frac{2}{x-3} + \frac{5}{x+1}$$

When decomposing a rational function you need to start by setting your function equal to the unknown fraction that you are trying to find. There are 3 different layouts, depending on the starting function:

Both (all) linear terms in the denominator:

$$\frac{7x-13}{(x-3)(x+1)} = \frac{\boldsymbol{A}}{\boldsymbol{(x-3)}} + \frac{\boldsymbol{B}}{\boldsymbol{(x+1)}} = \frac{A(x+1)+B(x-3)}{(x-3)(x+1)}$$

Repeated term in the denominator:

$$\frac{3x^2+7x-12}{(x-5)(x+2)^2} = \frac{\boldsymbol{A}}{\boldsymbol{(x-5)}} + \frac{\boldsymbol{B}}{\boldsymbol{(x+2)}} + \frac{\boldsymbol{C}}{\boldsymbol{(x+2)^2}} = \frac{A(x+2)^2+B(x-5)(x+2)+C(x-5)}{(x-5)(x+2)^2}$$

The second one (repeated term) can be tricky to combine. For each fraction, think about what the denominator has been multiplied by and do the same to the numerator.

Method for finding partial fractions:

Step 1: Set your functions equal to the correct (unknown) fraction, as above.

Step 2: Add the fractions (the same as you would with basic fractions) using a common denominator (this should always be identical to the original denominator). This step is shown in the 3rd column of the box above.

Step 3: The denominators are identical so you can ignore them and set the numerators equal.

Step 4: Substitute in sensible values of x and/or equate coefficients to create enough equations to find the values of A, B, C etc.

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e.g. decompose $\frac{7x^2+5x-42}{(x+4)(x-1)^2}$ into partial fractions.

The denominator has a repeated term so use the 2nd layout:

$$\frac{7x^2+5x-42}{(x+4)(x-1)^2} = \frac{A}{(x+4)}+\frac{B}{(x-1)}+\frac{C}{(x-1)^2} = \frac{A(x-1)^2+B(x+4)(x-1)+C(x+4)}{(x+4)(x-1)^2}$$

You have made sure that the denominators are equal so the numerators will also be equal:

$$7x^2+5x-42 = A(x-1)^2+B(x+4)(x-1)+C(x+4)$$

Now substitute in values of x that will help to find A, B and C. Remember that if you can make some terms =0, it will be easier to solve.

Let $x=1$: $\quad 7+5-42=0+0+C(1+4) \quad \rightarrow \quad -30=5C \quad \rightarrow \quad \boldsymbol{C=-6}$
Let $x=-4$: $\quad 112-20-42=A(-4-1)^2+0+0) \quad \rightarrow \quad 50=25A \quad \rightarrow \quad \boldsymbol{A=2}$

You can't choose any other values to make a term disappear so use any value of x to create another equation to find B. Just choose a small, easy value – often $x=0$ works well.

Let $x=0$: $\quad 0+0-42=A-4B+4C) \quad \rightarrow \quad -42=2-4B-24 \quad \rightarrow \quad \boldsymbol{B=5}$

You have found that $\frac{7x^2+5x-42}{(x+4)(x-1)^2}$ can be written as $\frac{2}{(x+4)}+\frac{5}{(x-1)}-\frac{6}{(x-1)^2}$