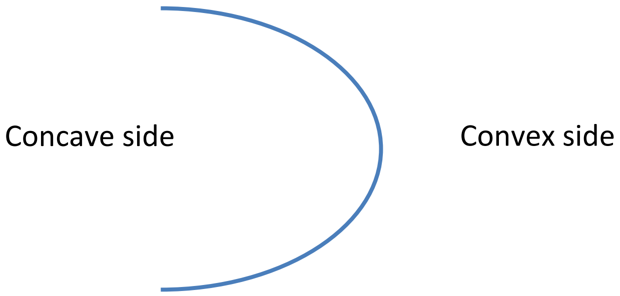
Summary sheet: Differentiation

G1 Understand and use the second derivative as the rate of change of gradient; connection to convex and concave sections of curves and points of inflection G4 Differentiate using the product rule, the quotient rule and the chain rule, including problems involving connected rates of change and inverse functions

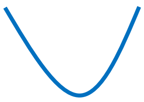
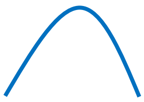
See AS summary sheet for a reminder of previous work.

Some definitions

Concave: Curves inwards (imagine you're standing inside a semi-circle – it curves towards you)
Convex: Curves outwards (now you're outside the semi-circle – it curves away from you)



Second derivative

Remember:	New terminology
If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\boldsymbol{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} > 0}$ it's a minimum	Increasing gradient **Concave upwards** (convex downwards) 
If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\boldsymbol{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} < 0}$ it's a maximum	Decreasing gradient **Concave downwards** (convex upwards) 
If $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ and $\boldsymbol{\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = 0}$ it could be a point of inflection or a maximum or minimum (you need further investigation to decide)	Where the curve changes from concave downwards to concave upwards or vice versa. 

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Product rule

Use the product rule to differentiate when 2 functions are multiplied together.

If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

OR

$(f(x)g(x))' = f(x)g'(x) + g(x)f'(x)$

Quotient rule

Use the quotient rule to differentiate a function divided by a function.

If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

OR

$\left(\frac{f(x)}{g(x)}\right)' = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$

For both the **product** and the **quotient** rule write down 4 pieces of information:

$u =$ $\qquad$ $v =$

$\frac{du}{dx} =$ $\qquad$ $\frac{dv}{dx} =$

Once you have these 4 things you can just substitute into the correct rule.

e.g. differentiate $y = x^2 \sin x$

$u = x^2$ $\qquad$ $v = \sin x$

$\frac{du}{dx} = 2x$ $\qquad$ $\frac{dv}{dx} = \cos x$

$\frac{dy}{dx} = x^2 \cos x + 2x \sin x$

Chain rule

Use the chain rule to differentiate a function of a function.

If $y = f(g(x))$, *let* $u = g(x)$ *then* $y = f(u)$ *and*

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

OR

$(f(g(x)))' = f'(g(x))g'(x)$

An easy way to do the chain rule is:

derivative of 'outside' function X derivative of 'inside' function

e.g. differentiate $y = (x^2 + 3)^7$

$$\frac{dy}{dx} = \underbrace{7(x^2+3)^6}_{\text{outside}} \times \underbrace{2x}_{\text{inside}}$$

$$\frac{dy}{dx} = 14x(x^2+3)^6$$

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Rates of change

If you are asked to find the **rate** of change of **something** you need to do:

$$\boxed{\frac{\mathrm{d}(\boldsymbol{something})}{\mathrm{d}t}}$$

Change in something

Change in time

For example: the rate of change of area (A) = $\frac{\mathrm{d}(A)}{\mathrm{d}t}$

the rate of change of height (h) = $\frac{\mathrm{d}(h)}{\mathrm{d}t}$

The chain rule

Following on from the chain rule, rates of change can be manipulated in the same way as normal fractions to find the one you want.

Examples:

You already have $\frac{\mathrm{d}A}{\mathrm{d}l}$ and $\frac{\mathrm{d}l}{\mathrm{d}t}$ and need to find $\boldsymbol{\frac{\mathrm{d}A}{\mathrm{d}t}}$: multiplying gives $\frac{\mathrm{d}A}{\mathrm{d}l} \times \frac{\mathrm{d}l}{\mathrm{d}t} = \boldsymbol{\frac{\mathrm{d}A}{\mathrm{d}t}}$

You already have $\frac{\mathrm{d}s}{\mathrm{d}r}$ and $\frac{\mathrm{d}t}{\mathrm{d}r}$ and need to find $\boldsymbol{\frac{\mathrm{d}s}{\mathrm{d}t}}$: dividing gives $\frac{\mathrm{d}s}{\mathrm{d}r} \div \frac{\mathrm{d}t}{\mathrm{d}r} = \frac{\mathrm{d}s}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = \boldsymbol{\frac{\mathrm{d}s}{\mathrm{d}t}}$

e.g. A water leak is forming a circular puddle that is gradually increasing in size. If the radius is increasing at a rate of 8cm/hr find the rate of increase of the area.

You are given: $\frac{\mathrm{d}r}{\mathrm{d}t} = 8$ and you want to find $\frac{\mathrm{d}A}{\mathrm{d}t}$

You know the area of a circle: $A = \pi r^2$

$$\therefore \frac{\mathrm{d}A}{\mathrm{d}r} = 2\pi r$$

Now use the two derivatives to find the one you want:

$$\frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}r} \times \frac{\mathrm{d}r}{\mathrm{d}t} = 2\pi r \times 8 = \boldsymbol{16\pi r}$$

Inverse functions

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}y}} \quad \text{and} \quad \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}$$

So if you are asked to find $\frac{\mathrm{d}x}{\mathrm{d}y}$ you can just find $\frac{\mathrm{d}y}{\mathrm{d}x}$ as normal and then "turn it upside down" $\left(i.e. do \frac{1}{\frac{\mathrm{d}y}{\mathrm{d}x}}\right)$

e.g. if $\boldsymbol{y = 3x^2 - 5x}$ find $\boldsymbol{\frac{\mathrm{d}x}{\mathrm{d}y}}$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x - 5$ $\qquad \therefore \frac{\mathrm{d}x}{\mathrm{d}y} = \frac{1}{6x-5}$