Summary sheet: Proof

As: Understand and use the structure of mathematical proof, proceeding from given assumptions through a series of logical steps to a conclusion; use methods of proof, including proof by deduction, proof by exhaustion. Disproof by counter example.
A: Proof by contradiction (including proof of the irrationality of √2 and the infinity of primes, and application to unfamiliar proofs).

Proof

A proof needs a mathematical argument to show that your theory (conjecture) is always true. You **cannot** just check lots of values and then state that something is true, because there might be a value, that you didn't try, that doesn't work.

Proof by deduction

Proof by deduction involves a set of logical steps. It is often useful to express what you are trying to prove, algebraically and then manipulate. Don't be afraid to put pen to paper and try things to see if they work.

e.g. Prove that the sum of any 3 consecutive integers is always divisible by 3.

Let the 1st number $= n$, so the 2nd number will be $n+1$ and the 3rd will be $n+2$

∴ the sum of the numbers $= n + n + 1 + n + 2$
$= 3n + 3$
factorises to : $3(n+1)$

which is always divisible by 3.

Proof by exhaustion

You can use this method when you are sure that all the possibilities can be tested. For example – if you were asked to prove that all the neighbours on your street could answer a basic maths question – you could walk around and ask them all.

e.g. Prove that no square number ends in 2, 3, 7 or 8.

You cannot test every square number but you can test the final digit. If you think about squaring a number the last digit of the answer comes from squaring the last digit of the original. E.g. $27^2 = 729$, the last digit (9) came from squaring the 7 $(7^2 = 49)$. So you can just consider all the single digits.

$0^2 = 0$	$5^2 = 25$
$1^2 = 1$	$6^2 = 36$
$2^2 = 4$	$7^2 = 49$
$3^2 = 9$	$8^2 = 64$
$4^2 = 16$	$9^2 = 81$

You have tried all the possibilities for square numbers and none of them end in 2, 3, 7 or 8.

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Disproof by counter example

If you can find a single instance where your conjecture is not true – this is a counter example.

e.g. it is suggested that $n^2 + n + 5$ is prime for all values of n.

Try some values:

Let	$n = 1 \rightarrow$	$1^2 + 1 + 5 = 7$	prime
	$n = 2 \rightarrow$	$2^2 + 2 + 5 = 11$	prime
	$n = 3 \rightarrow$	$3^2 + 3 + 5 = 17$	prime
	$n = 4 \rightarrow$	$4^2 + 4 + 5 = 25$	not prime

You have found a counter example to disprove the theory.

Proof by contradiction

You start off by assuming that your conjecture is false (i.e. the opposite is true) – then follow logical steps until you contradict yourself. You have proved it must be true.

e.g. Proof of the irrationality of $\sqrt{2}$

Remember that irrational means a number that cannot be written as a fraction (i.e. the ratio of 2 integers). This also means that, as a decimal it doesn't end, or repeat.

1st assume that $\sqrt{2}$ **is** rational and could be written as $\frac{a}{b}$ where a and b are integers (and $b \neq 0$)

You also assume that $\frac{a}{b}$ is simplified (cancelled) to its lowest terms. This means that a and b cannot both be even numbers (because you would have then simplified further).

Assume that $\sqrt{2}$ **is** rational: $\sqrt{2} = \frac{a}{b}$

$$\therefore 2 = \left(\frac{a}{b}\right)^2$$

$$2 = \frac{a^2}{b^2}$$

$$\therefore 2b^2 = a^2$$

This tells you that a^2 is even (i.e. it's 2 x something).
$\therefore$ ***a* is even** (it can't be odd because odd^2 = odd)

Now you know that a is even, replace it with $2k$

$$\therefore 2b^2 = (2k)^2$$

$$\therefore 2b^2 = 4k^2$$

$$\therefore b^2 = 2k^2$$

This tells you that b^2 is even
$\therefore$ ***b* is even**

Now you have a contradiction because you started by saying that a and b cannot both be even. The contradiction means that your original assumption (that $\sqrt{2}$ **is** rational) is **not** correct. Therefore you have proved that $\sqrt{2}$ is irrational.

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e.g. Proof that there are infinitely many primes
Remember that any non-prime number is divisible by, at least, one prime.

Start by assuming that there are a **finite** number of primes.

You could list the primes as: $p_1, p_2, p_3, \ldots \ldots \ldots . p_n$ (where n is finite)
Now think of a new number, q where: $q = p_1 \times p_2 \times p_3 \times \ldots \ldots \times p_n + 1$

q is bigger than any of the primes, so it is not equal to any of them and $\therefore$ can't be prime.

Since q can't be prime it must be divisible by at least one prime, BUT when you divide q by any prime you get a remainder of 1.

This is a contradiction, so your original assumption (that there are a finite number of primes) is **not** correct. Therefore you have proved that there are infinitely many primes.