# Section 1: The general binomial expansion

## Notes and Examples

These notes contain subsections on

- The general binomial expansion
- Harder examples
- Finding approximations

## The general binomial expansion

You have already met the binomial expansion, which can be used to expand the expression $(1+x)^n$, where $n$ is a positive integer:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots$$

When $n$ is a positive integer, this expansion always has a finite number of terms, since eventually there will be a factor of 0 in the numerator.

However, this expansion can also be used for any value of $n$, including negative numbers and fractions. In cases where $n$ is not a positive integer, there will never be a zero coefficient, so the expansion will have an infinite number of terms.


This means that the expansion is only valid for cases where values of $x^r$ decrease as $r$ increases, i.e. for $-1 < x < 1$. Otherwise, the terms of the expansion would get bigger and bigger and the sum of the terms would increase without limit.

**Example 1**

Expand $(1+x)^{-3}$ up to the term in $x^3$, stating the values of $x$ for which the expansion is valid.

**Solution**

$$\begin{aligned}(1+x)^{-3} &= 1 - 3x + \frac{-3\times -4}{1\times 2}x^2 + \frac{-3\times -4\times -5}{1\times 2\times 3}x^3 + \ldots \\ &= 1 - 3x + 6x^2 - 10x^3 + \ldots\end{aligned}$$

The expansion is valid for $-1 < x < 1$.

The expansion still works when the second term in the bracket is a multiple of $x$. You just need to replace $x$ in the expansion with the multiple of $x$, and remember that, for example, $(2x)^3$ is not $2x^3$ but $8x^3$.

Also, be careful with the range of $x$ for which the expansion is valid in such cases. Look carefully at Example 2 below.

**Example 2**

Expand $\sqrt{1+2x}$ up to the term in $x^3$, stating the values of $x$ for which the expansion is valid.

**Solution**

$$\sqrt{1+2x} = (1+2x)^{\frac{1}{2}}$$

$$= 1 + \tfrac{1}{2}\times 2x + \frac{\frac{1}{2}\times -\frac{1}{2}}{1\times 2}(2x)^2 + \frac{\frac{1}{2}\times -\frac{1}{2}\times -\frac{3}{2}}{1\times 2\times 3}(2x)^3 + \dots$$

$$= 1 + x - \tfrac{1}{8}\times 4x^2 + \tfrac{1}{16}\times 8x^3 + \dots$$

$$= 1 + x - \tfrac{1}{2}x^2 + \tfrac{1}{2}x^3 + \dots$$

The expansion is valid for $-1 < 2x < 1$

i.e. $-\tfrac{1}{2} < x < \tfrac{1}{2}$.

When the second term involves a negative, be very careful with signs.

**Example 3**

Find the first four terms in the expansion of $\dfrac{1}{1-x}$, stating the values of $x$ for which the expansion is valid.

**Solution**

$$\frac{1}{1-x} = (1-x)^{-1}$$

$$= 1 - 1(-x) + \frac{-1\times -2}{1\times 2}(-x)^2 + \frac{-1\times -2\times -3}{1\times 2\times 3}(-x)^3 + \dots$$

$$= 1 + x + x^2 + x^3 + \dots$$

The expansion is valid for $-1 < x < 1$.

## Harder examples

You can expand expressions like $(a+b)^n$ by taking out a factor $a^n$ first like this:

$$(a+b)^n = a^n\left(1+\frac{b}{a}\right)^n.$$

You can then expand the expression in the bracket using the binomial expansion. Be very careful – it is easy to make mistakes in this kind of work.

This method is shown in the next example.

**Example 4**

Find the first four terms in the expansion of $\sqrt{4+x}$, stating the values of $x$ for which the expansion is valid.

**Solution**

$$\sqrt{4+x} = (4+x)^{\frac{1}{2}} = 4^{\frac{1}{2}}\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{x}{4}\right)^{\frac{1}{2}}$$

$$\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 1+\frac{1}{2}\left(\frac{x}{4}\right)+\frac{\frac{1}{2}\times-\frac{1}{2}}{1\times2}\left(\frac{x}{4}\right)^2+\frac{\frac{1}{2}\times-\frac{1}{2}\times-\frac{3}{2}}{1\times2\times3}\left(\frac{x}{4}\right)^3+\ldots$$

$$= 1+\frac{x}{8}-\left(\frac{1}{8}\times\frac{x^2}{16}\right)+\left(\frac{1}{16}\times\frac{x^3}{64}\right)+\ldots$$

$$= 1+\frac{x}{8}-\frac{x^2}{128}+\frac{x^3}{1024}+\ldots$$

$$2\left(1+\frac{x}{4}\right)^{\frac{1}{2}} = 2\left(1+\frac{x}{8}-\frac{x^2}{128}+\frac{x^3}{1024}+\ldots\right)$$

$$= 2+\frac{x}{4}-\frac{x^2}{64}+\frac{x^3}{512}+\ldots$$

The expansion is valid for $-1<\frac{1}{4}x<1$

i.e. $-4<x<4$

The next example shows a more complicated expansion.

**Example 5**

Find the first three terms of the expansion of $\sqrt{\frac{1+2x}{1-x}}$, stating the values of $x$ for which the expansion is valid.

**Solution**

$$\sqrt{\frac{1+2x}{1-x}} = (1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}}$$

$$(1+2x)^{\frac{1}{2}} = 1+\tfrac{1}{2}\times2x+\frac{\frac{1}{2}\times-\frac{1}{2}}{1\times2}(2x)^2+\ldots$$

$$= 1+x-\tfrac{1}{8}\times4x^2+\ldots$$

$$= 1+x-\tfrac{1}{2}x^2+\ldots$$

$$(1-x)^{-\frac{1}{2}} = 1 - \tfrac{1}{2}\times(-x) + \frac{-\frac{1}{2}\times-\frac{3}{2}}{1\times 2}(-x)^2 + \ldots$$
$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \ldots$$

$$(1+2x)^{\frac{1}{2}}(1-x)^{-\frac{1}{2}} = \left(1 + x - \tfrac{1}{2}x^2 + \ldots\right)\left(1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + \ldots\right)$$
$$= 1\left(1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2\right) + x\left(1 + \tfrac{1}{2}x\right) - \tfrac{1}{2}x^2(1) + \ldots$$
$$= 1 + \tfrac{1}{2}x + \tfrac{3}{8}x^2 + x + \tfrac{1}{2}x^2 - \tfrac{1}{2}x^2 + \ldots$$
$$= 1 + \tfrac{3}{2}x + \tfrac{3}{8}x^2 + \ldots$$

When multiplying out the brackets, ignore any terms in powers of $x$ greater than $x^2$.

The expansion for $(1+2x)^{\frac{1}{2}}$ is valid for $-1 < 2x < 1$

i.e. $-\frac{1}{2} < x < \frac{1}{2}$

The expansion for $(1-x)^{-\frac{1}{2}}$ is valid for $-1 < x < 1$

For both conditions to be true, then $x$ must satisfy the condition $-\frac{1}{2} < x < \frac{1}{2}$.

## Finding approximations

The binomial expansion can be used for finding the approximate value of a function, by substituting an appropriate value for $x$ and evaluating the first few terms of the expansion. The more terms are used, the better the approximation.

**Example 6**

(i) Find the first four terms of the binomial expansion of $\sqrt[3]{(8+x)}$.

(ii) Use your result from (i) to find the value of $\sqrt[3]{8.1}$ correct to six decimal places.

**Solution**

(i) $\sqrt[3]{(8+x)} = (8+x)^{\frac{1}{3}} = 8^{\frac{1}{3}}\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 2\left(1+\frac{x}{8}\right)^{\frac{1}{3}}$

$$\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 1 + \frac{1}{3}\left(\frac{x}{8}\right) + \frac{\frac{1}{3}\times-\frac{2}{3}}{2!}\left(\frac{x}{8}\right)^2 + \frac{\frac{1}{3}\times-\frac{2}{3}\times-\frac{5}{3}}{3!}\left(\frac{x}{8}\right)^3 + \ldots$$
$$= 1 + \frac{x}{24} - \frac{x^2}{576} + \frac{5x^3}{41472} + \ldots$$

$$2\left(1+\frac{x}{8}\right)^{\frac{1}{3}} = 2\left(1 + \frac{x}{24} - \frac{x^2}{576} + \frac{5x^3}{41472} + \ldots\right)$$
$$= 2 + \frac{x}{12} - \frac{x^2}{288} + \frac{5x^3}{20736} + \ldots$$

(ii) Let $x = 0.1$

$$\sqrt[3]{(8+0.1} = 2 + \frac{0.1}{12} - \frac{0.1^2}{288} + \frac{0.1^3}{20736} + \ldots$$

Using the first two terms $\sqrt[3]{8.1} \approx 2 + \frac{0.1}{12} = 2.0083333$

Using the first three terms $\sqrt[3]{8.1} \approx 2 + \frac{0.1}{12} - \frac{0.1^2}{288} = 2.0082986$

Using the first four terms $\sqrt[3]{8.1} \approx 2 + \frac{0.1}{12} - \frac{0.1^2}{288} + \frac{5 \times 0.1^3}{20736} = 2.0082988$

Adding the fourth term does not affect the sixth decimal place
so $\sqrt[3]{8.1} = 2.008299$ correct to six decimal places.