Section 3: Geometric sequences and series

Notes and Examples

These notes contain subsections on

Formulae for geometric sequences

In the section on Arithmetic sequences, you used two basic formulae to solve problems: the formula for the kth term in an arithmetic sequence, and the formula for the sum of the first n terms of the sequence.

When you are working with geometric sequences, you need to use the equivalent formulae, and you may also need to use a third formula, for the sum to infinity of the sequence.

The formula for the kth term of a geometric sequence is

$$a_k = ar^{k-1}$$

where a is the first term of the sequence and r is the common ratio (the number that each term is multiplied by to obtain the next term).

The formula for the sum of the first n terms of a geometric sequence is

$$S_n = \frac{a(1-r^n)}{1-r} \quad \text{or} \quad S_n = \frac{a(r^n-1)}{r-1}$$

These two formulae are equivalent. If r is less than 1, then it is easier to use the left-hand formula, and if r is greater than 1, it is easier to use the right-hand version. However, either will give you the right answer!

The formula for the sum to infinity of a geometric sequence is

$$S_\infty = \frac{a}{1-r} \quad \text{for } -1 < r < 1$$

The restriction $-1 < r < 1$ is very important, as it is only for these values of r that the series *converges* (i.e. the terms become numerically smaller and smaller). If $r > 1$, the terms become larger and larger and so the sum of the terms becomes larger and larger, so it makes no sense to talk about the sum to infinity. If $r < -1$, the terms become numerically larger and larger, but alternate between positive and negative, so the sum of the terms becomes alternately large and small, and again it makes no sense to talk about the sum to infinity. In both the cases $r > 1$ and $r < -1$, the series *diverges*.

Worked examples

Example 1

A geometric sequence has first term 4 and common ratio ½.

(i) Find the 6th term.

(ii) Find the sum of the first 10 terms.

(iii) Find the sum to infinity of the terms of the sequence.

Solution

(i) $a_k = ar^{k-1}$

6^{th} term $= 4 \times \frac{1}{2}^5 = 0.125$

Substituting a = 4, r = ½ and k = 6

(ii) $S_n = \dfrac{a(1-r^n)}{1-r}$

$S_{10} = \dfrac{4(1-\frac{1}{2}^{10})}{1-\frac{1}{2}} = 8(1-\frac{1}{2}^{10}) = 7.992$

Substituting a = 4, r = ½ and n = 10

(iii) $S_\infty = \dfrac{a}{1-r}$

$= \dfrac{4}{1-\frac{1}{2}}$

$= 8$

Substituting a = 4, and r = ½

Example 2

A geometric series has common ratio 0.5 and the sum of the first 5 terms is 77.5. Find the first term.

Solution

$S_n = \dfrac{a(1-r^n)}{1-r}$

$77.5 = \dfrac{a\left(1-0.5^5\right)}{(1-0.5)}$

$77.5 = \dfrac{0.96875a}{0.5}$

$a = 40$

The first term is 40.

Substituting r = 0.5, S_n = 77.5 and n = 5

In Example 3, the common ratio is negative. Be careful when you are using your calculator to work out powers of a negative number – you will probably need to use brackets. If you type in –2^8 then the calculator will probably work out 2^8 and then apply a negative. So you need to type in (-2)^8. Remember that even powers of a negative number are positive, and odd powers are negative, and check that your answer is sensible.

Example 3

A geometric sequence has 3rd term 12 and 6th term –96.

(i) Find the first term and the common ratio.

(ii) Find the sum of the first 8 terms.

Solution

(i) 3rd term = $ar^2 \quad \Rightarrow ar^2 = 12$

6th term = $ar^5 \quad \Rightarrow ar^5 = -96$

Dividing the second equation by the first gives $r^3 = -8$

$r = -2$

$ar^2 = 12 \Rightarrow 4a = 12 \Rightarrow a = 3$

The first term is 3 and the common ratio is –2.

(ii) $S_n = \dfrac{a(r^n - 1)}{r - 1}$

$$S_8 = \frac{3((-2)^8 - 1)}{-2 - 1}$$
$$= \frac{3(256 - 1)}{-3}$$
$$= -255$$

In the next example, you have to use the formula for the sum of an infinite geometric series.

Example 4

(i) The sum of an infinite geometric series with common ratio -0.25 is 6.4. Find the first term.

(ii) The sum of an infinite geometric series with first term 3 is 5. Find the common ratio.

Solution

(i) $S_\infty = \dfrac{a}{1 - r}$

$$6.4 = \frac{a}{1 - (-0.25)}$$
$$= \frac{a}{1.25}$$
$$a = 6.4 \times 1.25 = 8$$

The first term is 8.

(ii) $S_\infty = \dfrac{a}{1-r}$

$$5 = \frac{3}{1-r}$$

$$1-r = \frac{3}{5} = 0.6$$

$$r = 1-0.6 = 0.4$$

The common ratio is 0.4.

Finding the number of terms

In Examples 5 and 6, you have to find the value of n, which appears as an index. Sometimes you need to use logarithms to find n.

Example 5
The sum of a geometric series with first term 3 and common ratio -2 is 513.
Find the number of terms in the series.

Solution

$$S_n = \frac{a(1-r^n)}{1-r}$$

$$513 = \frac{3\left(1-(-2)^n\right)}{\left(1-(-2)\right)}$$

$$513 = \frac{3\left(1-(-2)^n\right)}{3}$$

$$513 = 1-(-2)^n$$

$$(-2)^n = -512$$

By trial and improvement $n = 9$.

There are 9 terms in the series.

In the next example you have to find the number of terms required to exceed a given sum.

Example 6
Carlos saves money every year. The first year he saves £100. Each year he increases the amount he saves by 10%. After how many years do Carlos's savings first exceed £1000 (excluding any interest he has earned)?

Solution
The amount Carlos saves each year forms a geometric sequence with $a = 100$ and

$r = 1.1$.
The sum of Carlos's savings after n years is given by the formula

$$S_n = \frac{100(1.1^n - 1)}{1.1 - 1}$$
$$= \frac{100(1.1^n - 1)}{0.1}$$
$$= 1000(1.1^n - 1)$$

When this amount exceeds 1000

$$1000(1.1^n - 1) > 1000$$
$$1.1^n - 1 > 1$$
$$1.1^n > 2$$
$$\log 1.1^n > \log 2$$
$$n \log 1.1 > \log 2$$
$$n > \frac{\log 2}{\log 1.1}$$
$$n > 7.27$$

The savings first exceed £1000 after 8 years.